# An Introduction to Topological Groups

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## Abstract

This project is a survey of topological groups. Specifically, our goal is to investigate properties and examples of locally compact topological groups.

Our project is structured as follows. In Chapter 2, we review the basics of topology and group theory that will be needed to understand topological groups. This summary includes definitions and examples of topologies and topological spaces, continuity, the product topology, homeomorphism, compactness and local compactness, normal subgroups and quotient groups. In Chapter 3, we discuss semitopological groups. This includes the left and right translations of a group $G$, the left and right embeddings of $G$, products of semitopological groups and compact semitopological groups. Chapter 4 is on topological groups, here we discuss subgroups, quotient groups, and products of topological groups. We end the project with locally compact topological groups; here we investigate compactness and local compactness in topological groups. An important class of locally compact topological groups are groups of matrices. These structures are important in physics, we go over some of their basic properties.

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CHAPTER 1

# Introduction

Topology is an umbrella term that includes several fields of study. These include point-set topology, algebraic topology, and differential topology. Because of this it is difficult to credit a single mathematician with introducing topology. The following mathematicians all made key contributions to the subject: Georg Cantor, David Hilbert, Felix Hausdorff, Maurice Fréchet, and Henri Poincaré.

In general, topology is a special kind of geometry, a geometry that doesn't include a notion of distance. Topology has many roots in graph theory. When Leonhard Euler was working on the famous Königsberg bridge problem he was developing a type of geometry that did not rely on distance, but rather how different points are connected. This idea is at the heart of topology.

A topological group is a set that has both a topological structure and an algebraic structure. In this project many interesting properties and examples of such objects will be explored.

CHAPTER 2

# Preliminaries

## 1. Topology

Just as a metric space is a generalization of a Euclidean space, a topological space is a generalization of a metric space. Instead of having a metric that tells us the distance between two points, topological spaces rely on a different notion of closeness; points are related by open sets. Our primary sources for this section are [**1**] and [**2**].

Definition 2.1. *Let $X$ be a set. A* **topology** *on $X$ is a collection $\tau$ of subsets of $X$ that satisfy the following three requirements:*

(1) $\emptyset \in \tau$ *and* $X \in \tau$
(2) *Given* $\mathscr{U} \subset \tau$, *we have* $\cup\{U : U \in \mathscr{U}\} \in \tau$ *(Closure under arbitrary unions)*
(3) *Given* $U_1$ *and* $U_2 \in \tau$, *we have* $U_1 \cap U_2 \in \tau$ *(Closure under finite intersections)*

*Members of a topology are called open sets.*

Example 2.2. For a set $X$ there are two topologies that arise commonly, the discrete and indiscrete topologies.

(1) When $\tau = \mathcal{P}(X)$, $\tau$ is called the discrete topology. Here $\mathcal{P}(X)$ is the notation for the power set of X: the set of all subsets of $X$,
(2) When $\tau = \{\emptyset, X\}$, $\tau$ is called the indiscrete topology.

Clearly these two examples satisfy the requirements of the previous definition.

Definition 2.3. *Let $X$ be a set and let $\tau_1$ and $\tau_2$ be topologies on $X$. If $\tau_1 \subset \tau_2$, then $\tau_1$ is* **coarser** *than $\tau_2$. If $\tau_2 \subset \tau_1$, then $\tau_1$ is* **finer** *than $\tau_2$.*

Definition 2.4. *The elements of the* **Euclidean topology** *on $\mathbb{R}^n$ are unions of open balls in $\mathbb{R}^n$. This topology is denoted by $\|\cdot\|_n$. A special case that we will make constant reference to is $\mathbb{R}$, here open sets are unions of open intervals.*

Definition 2.5. *A set $X$ together with a topology $\tau$ on $X$ form a* **topological space**. *This is denoted by the pair $(X, \tau)$.*

Definition 2.6. *Let $(X, \tau)$ be a topological space and $x \in X$. A set $N \subset X$ is a* **neighbourhood** *of $x$ if there exists some $U \in \tau$ with $x \in U \subset N$. In other words, $N$ is a set that contains an open set containing $x$. The set of all neighbourhoods of an element $x$ will be denoted by $\mathscr{N}_x$.*

Definition 2.7. *Let $(X,\tau)$ be a topological space. We say that a subset $F$ of $X$ is* **closed** *when $X \backslash F \in \tau$.*

Example 2.8. In $\mathbb{R}$ with the Euclidean topology, the set $[0,1]$ is closed. This is because $\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$, which is the union of two open intervals.

Example 2.9. In $(X, \mathcal{P}(X))$ every subset of $X$ is closed. This is the case because for any $F \subset X$ we have $X \setminus F \in \mathcal{P}(X)$.

Given a topological space $(X,\tau)$, subsets of $X$ can be: open, closed, both open and closed, or neither open nor closed. We will use the convention of calling sets that are both open and closed *clopen*. In the discrete space, every subset of $X$ is a clopen set.

Definition 2.10. *Let $(X,\tau)$ be a topological space and $S$ a subset of $X$. The* **closure** *of $S$, denoted $\bar{S}$, is defined to be $\bar{S} = \cap\{F : F \subset X$ is closed and contains $S\}$.*

In symbols the closure can be written as $\bar{S} = \{x \in X : N \cap S \neq \emptyset \text{ for all } N \in \mathscr{N}_x\}$. It isn't obvious that these two sets are the same, a proof of this is given in [**1**]. Intuitively, we can think of $\bar{S}$ as the smallest closed set that contains S.

Definition 2.11. *Let $(X,\tau)$ be a topological space. We say that a set $D \subset X$ is* **dense** *in $X$ when $\bar{D} = X$.*

Example 2.12. *$\mathbb{Q}$ is dense in $\mathbb{R}$ with the Euclidean topology.*

Definition 2.13. *Let $(X,\tau)$ be a topological space. A* **base** *$\mathscr{B}$ for $\tau$ is a subset of $\tau$ where each open set in $\tau$ can be written as a unions of elements in $\mathscr{B}$.*

Definition 2.14. *Let $(X,\tau)$ be a topological space. A* **subbase** *for $\tau$ is a set $\mathscr{S} \subset \tau$ with the property that the set of all finite intersections of sets in $\mathscr{S}$ is a base for $\tau$.*

Upon first seeing the definition for a base and subbase it can be difficult to understand exactly what is being said, since the two definitions are so similar. The following example should clarify the distinction between the two.

Example 2.15. Let $(X,\tau)$ be a topological space with $X = \mathbb{R}^3$ and $\tau$ is the Euclidean topology in $\mathbb{R}^3$. The set of all open boxes is a base for this topology since any open set can be written as a union of open boxes. A subbase for $\tau$ is the set of all open boxes with volume $\geq 1$. This is a subbase for $\tau$ since any open box can be written as a finite intersection of elements in the subbase. The subbase given is not a base for $\tau$ because we would be unable to generate any open sets of volume less than 1 since we can only take unions.

Definition 2.16. *A* **fundamental system of neighbourhoods** *can be thought of as localizing the notion of a base around a single point. Let $(X,\tau)$ be a topological space, let $x \in X$, and let $\mathscr{U}_x$ be the set of all open neighbourhoods of $x$. Let $\mathscr{V}_x$ be a subset of $\mathscr{U}_x$. We say that $\mathscr{V}_x$ is a fundamental system of neighbourhoods of $x$ when for all $U_x$ in $\mathscr{U}_x$ there exists some $V_x$ in $\mathscr{V}_x$ with $V_x \subset U_x$. We say $\mathscr{V}_x$ is a base for $\mathscr{U}_x$.*

Definition 2.17. *A topological space* $(X, \tau)$ *is called a* **Hausdorff** *space when distinct points can be separated by open sets. In symbols:* $\forall\, x, y \in X$ *with* $x \neq y$ $\exists\, U, V \in \tau$ *with* $x \in U$ *and* $y \in V$ *and* $U \cap V = \emptyset$.

Example 2.18. An example of a Hausdorff space is $\mathbb{R}$ with the Euclidean topology.

Proof. Let $x, y$ with $x \neq y$ be two elements of $\mathbb{R}$ and let $\epsilon = \frac{|x-y|}{3}$. It follows that $(x-\epsilon,\, x+\epsilon)$ and $(y-\epsilon,\, y+\epsilon)$ are disjoint open sets that contain $x$ and $y$, respectively. ☐

Definition 2.19. *Let* $(X, \tau_X)$ *and* $(Y, \tau_Y)$ *be topological spaces and let* $f : X \to Y$ *be a function. We call* $f$ **continuous** *at* $x_0 \in X$ *if the following holds:*
$N \in \mathscr{N}_{f(x_0)}$ *implies* $f^{-1}(N) \in \mathscr{N}_{x_0}$.

Proposition 2.20. *Let* $(X, \tau_X)$ *and* $(Y, \tau_Y)$ *be a topological spaces and let* $f : X \to Y$ *be a function. The following three statements are equivalent.*

(1) $f : X \to Y$ *is continuous on* $X$,
(2) $f^{-1}(U)$ *is open in* $X$ *for all open sets* $U$ *in* $Y$,
(3) $f^{-1}(F)$ *is closed in* $X$ *for all closed sets* $F$ *of* $Y$.

Many statements that are true for continuous functions in metric spaces are also true in topological spaces. An example of this is the transitivity of continuity.

Proposition 2.21. *Let* $(X, \tau_X)$ *and* $(Y, \tau_Y)$ *and* $(Z, \tau_Z)$ *be topological spaces such that* $g : X \to Y$ *is continuous at* $x_0 \in X$ *and* $f : Y \to Z$ *is continuous at* $g(x_0) \in Y$. *It follows that* $(f \circ g)(x)$ *is continuous at* $x_0$.

Proof. By Definition 2.20 we have:
$g : X \to Y$ is continuous at $x_0$: $\quad N \in \mathscr{N}_{g(x_0)}$ implies $g^{-1}(N) \in \mathscr{N}_{x_0}$
$f : Y \to Z$ is continuous at $g(x_0)$: $\quad M \in \mathscr{N}_{f(g(x_0))}$ implies $f^{-1}(M) \in \mathscr{N}_{g(x_0)}$
In order to prove that $(f \circ g)(x)$ is continuous at $x_0$ it must be shown that $N \in \mathscr{N}_{(f \circ g)(x_0)}$ implies $g^{-1}(f^{-1}(N)) \in \mathscr{N}_{x_0}$. Next, assume that $N \in \mathscr{N}_{(f(g(x_0))}$. Since $f$ is continuous at $g(x_0)$, it follows that $f^{-1}(N) \in \mathscr{N}_{g(x_0)}$. Since $g$ is continuous at $x_0$ we have $g^{-1}(f^{-1}(N)) \in \mathscr{N}_{x_0}$, as required. ☐

In order to discuss convergence in a topological space we need to define what a net is, as well as some related terms.

Definition 2.22. *A relation* $\preceq$ *on a set* $X$ *is an ordering when the following three requirements are satisfied. We refer to such sets as* **ordered sets**.

(1) $\forall\, \alpha \in X,\ \alpha \preceq \alpha$
(2) $\alpha \preceq \beta$ *and* $\beta \preceq \gamma$ *implies* $\alpha \preceq \gamma$
(3) $\alpha \preceq \beta$ *and* $\beta \preceq \alpha$ *implies* $\alpha = \beta$

Definition 2.23. *An ordered set* $X$ *is called* **directed** *when* $\alpha, \beta \in X$ *implies there exists* $\gamma \in X$ *with* $\alpha \preceq \gamma$ *and* $\beta \preceq \gamma$.

Definition 2.24. *A* **net** $(x_\alpha)_{\alpha\in\mathbb{A}}$ *in a set* $X$ *is a function from a directed set* $\mathbb{A}$ *to* $X$*. A net is a generalization of a sequence. We can take subnets of a net in exactly the same way we take subsequences of a given sequence.*

Definition 2.25. *Let* $(X, \tau)$ *be a topological space. We say that a net* $(x_\alpha)_{\alpha\in\mathbb{A}} \in X$ **converges** *to* $x \in X$ *when for all* $N \in \mathscr{N}_x$*, there is an* $\alpha_N \in \mathbb{A}$ *with* $x_\alpha \in N$ *for all* $\alpha \in \mathbb{A}$ *such that* $\alpha_N \preceq \alpha$*. This is denoted by* $x_\alpha \to x$ *(*$x$ *is the limit of the net).*

Earlier we defined the closure of a subset of $X$ in terms of closed sets and neighbourhoods. There is a different formulation of that concept we can build up with nets.

Proposition 2.26. *Let* $(X, \tau)$ *be a topological space and* $S$ *a subset of* $X$*. The* **closure** *of* $S$ *is the set of all points in* $X$ *that are a limit of a net in* $S$*. This can be worded as:* $x \in \bar{S}$ *if and only if* $x$ *is the limit of a net in* $S$*.*

This is a property of limits in metric spaces that carries over to topological spaces.

Proof. Consider $\mathscr{N}_x$ as a directed set defined by $M \preceq N$ when $N \subset M$ for $M, N \in \mathscr{N}_x$ for some $x$. Assume that $x \in \bar{S}$, by Definition 2.10 we have that for all $N \in \mathscr{N}_x$, there exists an $x_N \in N \cap S$. We see that the net $(x_N)_{N\in\mathscr{N}_x}$ converges to $x$, this is because all neighbourhoods of $x$ contain $x$. For the other direction let $(x_\alpha)_{\alpha\in\mathbb{A}}$ be a net in $S$ that converges to $x$. Assume by way of contradiction that $x \notin \bar{S}$, so $x \in X \setminus \bar{S}$. Recall that $\bar{S}$ is closed, this implies that $X \setminus \bar{S}$ is open, which means that $X \setminus \bar{S}$ is a neighbourhood of $x$. Since $(x_\alpha)_{\alpha\in\mathbb{A}}$ converges to a point outside of $\bar{S}$, there exists some $\alpha_N \in \mathbb{A}$ such that $x_\alpha \in X \setminus \bar{S}$ for all $\alpha_N \preceq \alpha$. We know that $x \setminus \bar{S} \subset x \setminus S$, so $x_\alpha \in X \setminus S$ for all $\alpha_N \preceq \alpha$. This is a contradiction since $(x_\alpha)_{\alpha\in\mathbb{A}}$ is a net in $S$.

□

Now for a useful result that immediately follows from the preceding proof.

Corollary 2.27. $F \subset X$ *is closed if and only if every net in* $F$ *that converges in* $X$ *has a limit in* $F$*.*

In metric spaces the limit of a sequence is unique. In general, this is a property that does not hold in topological spaces.

Proposition 2.28. *A topological space* $(X, \tau)$ *is Hausdorff if and only if every convergent net in the space has a unique limit.*

Example 2.29. Consider the space $(\mathbb{Z}, \{\emptyset, \mathbb{Z}\})$. In this space every net converges to every member of $\mathbb{Z}$. This is because the only neighbourhood of any point in $\mathbb{Z}$ is the set $\mathbb{Z}$ itself. This space is not Hausdorff.

A recurring theme in algebra and topology is how to construct new spaces from old ones. Two methods of doing so will now be discussed.

Definition 2.30. *Let $(X, \tau)$ be a topological space and let $Y \subset X$. We define $\tau|_Y = \{Y \cap U : U \in \tau\}$. It is clear that this set is a topology on $Y$. This topology is called the topology inherited from $X$.*

Definition 2.31. Let $((X_i, \tau_i))_{i \in \mathbb{I}}$ be a family of topological spaces and $(X, \tau)$ be the topological product. The functions $\pi_i : X \to X_i$ defined by $\pi_i\,(\,(x_1,\, x_2, \ldots x_i, \ldots.)\,) = x_i$ are called the **coordinate projections**. We can think of $\pi_i$ as picking out the $i^{th}$ component of $X$.

Definition 2.32. [**4**] *Let $((X_i, \tau_i))_{i \in \mathbb{I}}$ be a family of topological spaces and let $X = \prod_{i \in \mathbb{I}} X_i$ as sets. Depending on whether or not $\mathbb{I}$ is a finite set we can have two different topologies on $X$. If $\mathbb{I}$ is finite, then we use the box topology: $\tau = \left\{ \prod_{i \in \mathbb{I}} U_i \,:\, U_i \in \tau_i \right\}$. If $\mathbb{I}$ is an infinite set then the box topology would have too many elements to retain its desirable properties( Tychonoff's Theorem doesn't hold). Instead we define $\tau$ to be the coarsest topology that makes the coordinate projections $\pi_i : X \to X_i$ continuous. The product topology is generated from the basis $\left\{ \prod_{i \in \mathbb{I}} U_i \,:\, U_i \in \tau_i \, such\, that\, X_i \neq U_i \; for \; finitely \; many \; i \right\}$. If $\mathbb{I}$ is a finite set then the box and product topologies are the same. The pair $(X, \tau)$ is a called the* **topological product** *of $((X_i, \tau_i))_{i \in \mathbb{I}}$.*

Example 2.33. Consider $\mathbb{R}$ with the Euclidean topology, we can take $n$ copies of $\mathbb{R}$ and use the box topology to get $\mathbb{R}^n$. Since this is a finite product the box and product topologies yield the same result. In $\mathbb{R}^n$, $(\,(-1,1), (-2,2), (-3,3), \ldots, (-n,n)\,)$ is an example of an open set.

Next, we will discuss compactness in topological spaces.

Definition 2.34. *Let $(X, \tau)$ be a topological space and $S \subset X$. An* **open cover** *$\mathscr{O}$ of $S$ is a collection of open sets that contain $S$, in symbols we have $S \subset \cup\{U : U \in \mathscr{O}\}$.*

Definition 2.35. *Let $(X, \tau)$ be a topological space and $S \subset X$. The set $S$ is* **compact** *in $(X, \tau)$ when every open cover has a finite subcover. That is, for any collection $\mathscr{O}$ that covers $S$, there exist $U_1, U_2, \ldots U_n \in \mathscr{O}$ such that $S \subset \cup_{i=1}^{n} U_i$.*

Proposition 2.36. *Let $(X, \tau_1)$ be a compact space, let $(Y, \tau_2)$ be a topological space and let $f : X \to Y$ be a continuous function. Then $f(X)$ is compact.*

Proof. Let $\mathscr{O}$ be an open over for $f(X)$. By Proposition 2.20 we know that $\{f^{-1}(U) : U \in \mathscr{O}\}$ is an open cover for $X$. Since $X$ is compact, there is a finite subcover. There exist $U_1, U_2, \ldots U_n \in \mathscr{O}$ such that $X \subset f^{-1}(U_1) \cup f^{-1}(U_2) \cup \ldots \cup f^{-1}(U_n)$. It follows that $f(X) \subset U_1 \cup U_2 \cup \ldots \cup U_n$. Thus, an arbitrary open cover of $f(X)$ has a finite subcover. This completes the proof. □

Proposition 2.37. *Let $(X, \tau)$ be a compact topological space and let $Y$ be a closed subset of $X$. Then $Y$ is a compact set.*

Proof. Let $\mathscr{O}$ be an open cover for $Y$. Because $Y$ is closed in $X$, $X/Y$ is open in $X$. From this we get that $\mathscr{O} \cup X/Y$ is an open cover for $X$. Since $X$ is compact there must be a finite subcover. So, there exist $U_1, U_2, ... U_n \in \mathscr{U}$ such that $X \subset U_1 \cup U_2 \cup ... \cup U_n \cup (X/Y)$. It follows that $Y \subset U_1 \cup U_2 \cup ... \cup U_n$. An arbitrary open cover of $Y$ has a finite subcover, thus $Y$ is compact. $\square$

EXAMPLE 2.38. In any topological space a finite set is always compact.

THEOREM 2.39. **Heine-Borel Theorem**
*A subset of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

DEFINITION 2.40. *A topological space $(X, \tau)$ is* **locally compact** *when for each $x \in X$ there exists $U \in \tau$ with $x \in U$ such that $\bar{U}$ is compact.*

EXAMPLE 2.41. $\mathbb{R}^n$ with the Euclidean topology is locally compact.

PROPOSITION 2.42. *Let $(X, \tau)$ be a topological space. If $K_1$, $K_2$,... $K_n$ are a family of compact sets in $(X, \tau)$, then $\cup_{i=1}^n K_i$ is compact.*

PROOF. Let $\mathscr{O}$ be an open cover for $\cup_{i=1}^n K_i$. Since $K_i$ is a subset of $\cup_{i=1}^n K_i$ for all $i$, then $\mathscr{O}$ is an open cover for all $K_i$. Since each $K_i$ is compact, there is a finite subcover of $\mathscr{O}$ for each $i$, we denote each subcover by $\mathscr{O}_i$. It follows that $\cup_{i=1}^n \mathscr{O}_i$ covers $\cup_{i=1}^n K_i$. The finite union of a collection of finite sets in finite, and $\cup_{i=1}^n \mathscr{O}_i \subset \mathscr{O}$, so we have our finite subcover. This completes the proof. $\square$

DEFINITION 2.43. *Let $(X, \tau)$ be a topological space. We say $(X, \tau)$ has the* **finite intersection property** *when the following holds: Let $\mathcal{F}$ be a family of closed sets of $X$ with $\cap\{F : F \in \mathcal{F}\} = \emptyset$, then there exists a finite subfamily $F_1, F_2, ...F_n$ of elements of $\mathcal{F}$ such that $\cap_{i=1}^n F_i = \emptyset$.*

In some of the following proofs we will use a different, but equivalent, form of the finite intersection property: if, for all finite subfamilies of $\mathcal{F}$ we have $\cap_{i=1}^n F_i \neq \emptyset$, then $\cap\{F : F \in \mathcal{F}\} \neq \emptyset$.

PROPOSITION 2.44. *A topological space is compact if and only if it has the finite intersection property.*

THEOREM 2.45. **Tychonoff's Theorem**
*Let $((X_i, \tau_i))_{i \in \mathbb{I}}$ be a family of topological spaces with $X_i$ compact for all $i \in \mathbb{I}$. Then the topological product $(X, \tau)$ is compact.*

The proof of Tychonoff's Theorem is rather long and technical, see [**1**] for the details.

Tychonoff's theorem does not hold for locally compact spaces. Instead, we have the following theorem.

THEOREM 2.46. *A finite product of locally compact spaces is locally compact.*

DEFINITION 2.47. *A topological space $(X, \tau)$ is* **disconnected** *when there exist non-empty sets $U, V \in \tau$ such that $U \cap V = \emptyset$ and $U \cup V = X$. A set is* **connected** *when such sets don't exist.*

Definition 2.48. *In a topological space* $(X, \tau)$ *a* **component** *is a connected subspace that is not properly contained in any other connected subspace of* $X$*.*

Proposition 2.49. *Let* $(X, \tau)$ *be a topological space. If* $Y$ *is a dense connected subspace of* $X$*, then* $X$ *is connected.*

Proposition 2.50. *Given a topological space* $(X, \tau)$ *and a family* $\mathcal{Y}$ *of connected subspaces of* $X$ *where no two elements of* $\mathcal{Y}$ *are disjoint, it follows that* $\cup\{Y : Y \in \mathcal{Y}\}$ *is connected.*

Definition 2.51. *Let* $(X, \tau_X)$ *and* $(Y, \tau_Y)$ *be topological spaces. A bijection* $f : X \to Y$ *is called a* **homeomorphism** *when both* $f$ *and* $f^{-1}$ *are continuous.*

When two spaces are homeomorphic it means that they share all the same topological properties; they are topologically indistinguishable. This is the topological analog of an isomorphism of groups.

Example 2.52. Consider the sets [0,2] and [0,7] each with the discrete topology. The function given by $f(x) = \frac{7}{2}x$ is a homeomorphism. It is clear that $f$ is a continuous bijection. Taking the inverse we get $f^{-1}(x) = \frac{2}{7}x$, which is also continuous. Thus $f$ is a homeomorphism and [0,2] and [0,7] are homeomorphic

Example 2.53. Consider the sets [0,1] and (0,1) each with the Euclidean topology. These spaces are not homeomorphic since [0,1] is compact and (0,1) is not. This result follows from the Heine-Borel Theorem. Since one space exhibits a topological property that is absent in the other, the two cannot possibly be homeomorphic.

Example 2.54. Perhaps the most famous example of a homeomorphism in topology is between a coffee cup and a doughnut. There exists a function that maps every point on the coffee cup to a point on the doughnut in a continuous fashion.

We end this section by introducing the separation axioms. These axioms help us to characterize topological spaces.

Definition 2.55. *Let* $(X, \tau)$ *be a topological space. We call* $(X, \tau)$ *a* $T_0 -$ *space when the following holds. For all* $x, y \in X$ *with* $x \neq y$*, there exists* $U \in \tau$ *such that either* $x \in U$ *and* $y \notin U$*, or* $x \notin U$ *and* $y \in U$*.*

Definition 2.56. *Let* $(X, \tau)$ *be a topological space. We call* $(X, \tau)$ *a* $T_1 -$ *space when the following holds. For all* $x, y \in X$ *with* $x \neq y$*, there exist* $U, V \in \tau$ *such that* $x \in U$*,* $y \notin U$*,* $y \in V$*, and* $x \notin V$*.*

A $T_2 -$ *space* is a Hausdorff space, which we have defined earlier.

## 2. Group Theory

In this section we will review the basic tenets of group theory and common examples of groups. Our reference for this section is [**3**].

Definition 2.57. *Let $G$ be a nonempty set and let $* : G \times G \to G$ be a binary operation defined by $*(g_1, g_2) = (g_1 * g_2)$. The pair $(G, *)$ is a* **group** *if the following three properties hold:*

1. *For all $a, b, c \in G$ we have $(a * b) * c = a * (b * c)$ (Associativity)*
2. *There exists an $e \in G$ such that for all $a \in G$ we have $a * e = e * a = a$ (Identity Element)*
3. *For all $a \in G$ there exists $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$ (Inverse Elements)*

Definition 2.58. *Let $(G, *)$ be a group. If $G$ has the property that $a * b = b * a$ for all a,b $\in G$, then we call $G$* **abelian**.

Example 2.59. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ all form abelian groups under regular addition. The identity element in each case is 0 and the inverse of $a$ is $-a$.

Example 2.60. The sets $\mathbb{Z} \setminus \{0\}$, $\mathbb{Q} \setminus \{0\}$, $\mathbb{R} \setminus \{0\}$, $\mathbb{C} \setminus \{0\}$ all form abelian groups under multiplication. The identity element in each case is 1 and the inverse of $a$ is $\frac{1}{a}$.

Example 2.61. Let $\mathbb{F}$ be a field. We define $GL_n(\mathbb{F})$ to be the group of all $n \times n$ matrices with entries in $\mathbb{F}$ with non-zero determinants. This is a group under matrix multiplication. The identity element is the matrix with diagonal entries of 1 and 0 elsewhere. This is an example of a non-abelian group.

Example 2.62. *Let $\mathbb{F}$ be a field. We define $M_n(\mathbb{F})$ to be the group of all $n \times n$ matricies with entries in $\mathbb{F}$. The identity element is the matrix with all zero entries. This is an abelian group under addition.*

Example 2.63. *Let $S_n$ be the set of all permutations of the set of integers from 1 to $n$. With the composition operation $(S_n, \circ)$ is a group. The identity element is the permutation that sends every element to itself.*

Definition 2.64. *Let $(G, *)$ be a group and $H$ a subset of $G$. We call $H$ a* **subgroup** *of $G$ when the following holds:*

1. *$H \neq \emptyset$,*
2. *If $x, y \in H$, then $x * y \in H$ (Closure under the operation of G),*
3. *If $x \in H$, then $x^{-1} \in H$ (Closure under inverses).*

*We write $H \leq G$ to denote subgroups.*

Definition 2.65. *Let $G$ be a group, $H$ a subgroup of $H$ and $g \in G$. The sets $gH = \{g * h \,|\, h \in H\}$ and $Hg = \{h * g \,|\, h \in H\}$ are called the* **left** *and* **right cosets** *of $H$ in $G$.*

Definition 2.66. *A subgroup $H$ of a group $G$ is called* **normal** *when $gH = Hg$ for all $g \in G$. Normal subgroups are denoted by $H \trianglelefteq G$.*

Definition 2.67. *Let $G$ be a group and $H$ a normal subgroup of $G$. The* **quotient group** *$G/H$ is defined to be the set of all left cosets of $H$ in $G$, in symbols: $G/H = \{gH \mid g \in G\}$. The operation on $G/H$ is given by $(xH) * (yH) = (x * y)H$.*

Definition 2.68. *Let $(G, *_1)$ and $(H, *_2)$ be groups. A function $\phi : G \to H$ is a* **homomorphism** *when $\phi(x *_1 y) = \phi(x) *_2 \phi(y)$ holds for all $x, y \in G$.*

Definition 2.69. *Let $(G, *_1)$ and $(H, *_2)$ be groups and $\phi : G \to H$ a function. Then, $\phi$ is an* **isomorphism** *when $\phi$ is a bijective homomorphism. When two groups are isomorphic it means that they are indistinguishable as groups; every group theoretic property of $G$ is also a property of $H$.*

Definition 2.70. *The* **centre** *of a group $G$ is the set of all elements in $G$ that commute with all elements of $G$, denoted $Z(G)$. In symbols $Z(G) = \{g \in G : g*x = x*g \ \forall x \in G\}$.*

CHAPTER 3

# Semitopological Groups

Before studying topological groups we begin with a related structure, that of a semitopological group. Unless otherwise noted, definitions and proofs for this section come from [**2**].

Definition 3.1. *We call a triple $(G, \tau, *)$ a* **semitopological group** *when $(G, \tau)$ is a topological space and $(G, *)$ is a group, and the group operation $* : G \times G \to G$ that maps $(x, y)$ to $x * y$ is continuous in each variable separately. When there is no ambiguity as to what the operation and topology are, we will simply use $G$ to denote a semitopological group.*

The function $* : G \times G \to G$ is continuous in the variable $x$ when the function

$$g_{y_0} : G \to G \text{ defined by } x \to x * y_0$$

is continuous for all $y_0$ in $G$. Similarly, $*$ is continuous in $y$ when the function

$$g_{x_0} : G \to G \text{ defined by } y \to x_0 * y$$

is continuous for all $x_0$ in $G$.

Definition 3.2. *For $G$ to be a* **topological group** *we require $G$ to satisfy all of the conditions for a semitopological group as well as two more requirements. The group operation needs to be continuous in both variables together and the inverse mapping given by $x \to x^{-1}$ needs to be continuous.*

This definition of continuity in one variable may not be ideal to work with in a topological setting. An equivalent statement in terms of neighbourhoods will now be presented.

Proposition 3.3. *Consider the function $\psi : G \times G \to G$ that maps $(x, y)$ to $x * y$. This function is continuous in $x$ when, for all $W \in \mathscr{N}_{x*y_0}$, there exists $U \in \mathscr{N}_x$ and $y_0 \in G$ such that $Uy_0 \subset W$. Continuity in $y$ follows in a similar way.*

Proposition 3.4. *The function $\psi : G \times G \to G$ that maps $(x, y)$ to $x * y$ is continuous in both variables when for all $W \in \mathscr{N}_{x*y_0}$, there exists $U \in \mathscr{N}_x$ and $V \in \mathscr{N}_y$ such that $UV \subset W$.*

This follows directly from the definition of continuity. For any neighbourhood $W$ of $x * y_0$, there is $U$ such that $x \in U$ with $\psi(U) \subset W$. From this we get $Uy_0 \subset W$.

Any group with the discrete topology is both a topological group and a semitopological group.

EXAMPLE 3.5. $(S_3, \mathcal{P}(S_3), \circ)$ *is both a semitopological group and a topological group.*

PROPOSITION 3.6. *Let $G$ be a semitopological group and $a \in G$. The following functions from $G$ to $G$ are called the right and left translations of $G$ by $a$: $r_a : x \to xa$, $l_a : x \to ax$. The translations of $G$ yield a homeomorphism in each case.*

PROOF. The statement will be proved for $r_a$, the other translation follows in a similar manner. First we prove that $r_a$ is a bijection. Assume that $y \in G$, then the element $ya^{-1}$ maps to $y$, so $r_a$ is surjective. Assume that $r_a(x) = r_a(y)$, which means that $xa = ya$. We multiply by $a^{-1}$ on the right to get $x = y$, so $r_a$ is injective. Now to prove that $r_a$ and its inverse are continuous. Since $G$ is a semitopological group that has a continuous group operation in each variable separately, we know that $\forall\, W \in \mathcal{N}_{xy}\; \exists\, U \in \mathcal{N}_x$ with $U_y \subset W$. The function $r_a$ fixes an $a \in G$, so we get $\forall\, W \in \mathcal{N}_{xa}\; \exists\, U \in \mathcal{N}_x$ with $U_a \subset W$, which makes $r_a$ continuous. Consider $r_a^{-1}(x)$ which maps $xa$ to $x$, this is equivalent to the map from $x$ to $xa^{-1}$, so $r_a^{-1}(x) = r_{a^{-1}}(x)$. We have that $r_{a^{-1}}(x)$ is continuous by the same argument as above. So $r_a$ is a homeomorphism. $\square$

An immediate and useful result we get from this proposition is as follows:

COROLLARY 3.7. *Let $G$ be a semitopological group and $F$ be a closed in $G$, $U$ open in $G$, $A$ any subset of $G$, and $a \in G$. We have:*

(1) *$Fa$ and $aF$ are closed in $G$*
(2) *$Ua$, $aU$, $AU$, $UA$ are open in $G$.*

COROLLARY 3.8. *Let $G$ be a semitopological group and $x_1 x_2 \in G$. There exists a homeomorphism $f$ such that $f(x_1) = x_2$.*

PROOF. From Proposition 3.6 we know that $r_a$ is a homeomorphism for all $a \in G$. By letting $f = r_{x_1^{-1}x_2}$ we get $f(x_1) = x_2$ as required. $\square$

PROPOSITION 3.9. *Let $(G_i)_{i\in\mathbb{I}}$ be a family of semitopological groups. It follows that $G = \prod_{i\in\mathbb{I}} G_i$ with the product topology is a semitopological group.*

THEOREM 3.10. *Let $(G_i)_{i\in\mathbb{I}}$ be a family of compact semitopological groups. It follows that $G = \prod_{i\in\mathbb{I}} G_i$ is a compact semitopological group.*

PROOF. By the previous proposition we have that $\prod_{i\in\mathbb{I}} G_i$ is a semitopological group. Tychonoff's Theorem tells us that $\prod_{i\in\mathbb{I}} G_i$ is compact. $\square$

Next we will discuss embeddings of a group.

DEFINITION 3.11. *Given that for all $i \in \mathbb{I}$ we have $G_i = G$, we denote the product $\prod_{i\in\mathbb{I}} G_i = G \times G \times G ... \times G$ as $G^{\mathbb{I}}$. We can view $G^{\mathbb{I}}$ as the set of all functions from $\mathbb{I}$ to $G$. Consider the element $s \in G^{\mathbb{I}}$ given by $s = (x_{\alpha_1}, x_{\alpha_2}, ... x_{\alpha_j} ...)$ we can associate this element with the function $f_s : \mathbb{I} \to G^{\mathbb{I}}$ that maps $\alpha_i$ to $x_{\alpha_i}$ for all $i \in \mathbb{I}$.*

Proposition 3.12. *Let $G$ be a semitopological group. There exists a bijection from $G$ onto a subset of $G^G$. We call such a function an* **embedding** *of $G$.*

Proof. Define functions from $G$ to $G^G$ by $\eta_r : a \to r_a$ and $\eta_l : a \to l_a$. In this proof we will use $\eta_r$ but the left translations also work. First, it needs to be shown that $\eta_r$ is injective, so assume that $\eta_r(x) = \eta_r(y)$ for some $x, y = G$. It follows that $r_x = r_y$. Recall that $r_x$ and $r_y$ are the functions that map $a \to ax$ and $a \to ay$, respectively. If $r_x = r_y$, then $ax = ay$ for all $a \in G$, multiplying by $a^{-1}$ on the left gives us $x = y$ as required. So $\eta_r : G \to \eta_r(G)$ is a bijection. We call $\eta_r$ and $\eta_l$ the right and left **canonical embeddings** of $G$ into $G^G$. □

Theorem 3.13. *Let $G$ be a semitopological group. It follows that $G$ is homeomorphic with $\eta_r(G) \subset G^G$ with the product topology.*

Theorem 3.14. *A locally compact Hausdorff semitopological group with a group operation that is continuous in both variables together is a topological group.*

This theorem makes use of three lemmas. A full proof of each one is given in [**2**].

Lemma 3.15. *If $A$ is compact in $G$, then $A^{-1}$ is closed in $G$.*

Lemma 3.16. *If $x$ is a limit point of a countable subset $E$ of $G$, then $x^{-1}$ is a limit point of $E^{-1}$.*

Lemma 3.17. *If $A$ is compact in $G$, then $A^{-1}$ is compact in $G$.*

Proof. Let $G$ be as above. The only thing that needs to be proved is that the map $x \to x^{-1}$ is continuous. We will prove continuity at $e$, continuity everywhere follows from $r_a$ and $l_a$ being homeomorphisms. From the definition of continuity, we need to show that $\forall\, U \in \mathcal{N}_e\ \exists\, W \in \mathcal{N}_e$ such that $W^{-1} \subset U$. We will find a compact set $W$ that satisfies this inclusion. Let $\{K_i\}_{i\in\mathbb{I}}$ be the set of all compact neighbourhoods of $e$. Assume by way of contradiction that $K_i^{-1} \not\subset U$ for all $i \in \mathbb{I}$. From this we get $K_i^{-1} \cap (G \setminus U) \neq \emptyset$ for all $i \in \mathbb{I}$. Since $U$ is open, $(G \setminus U)$ is closed, this means that $K_i^{-1} \cap (G \setminus U)$ is a closed subset of $K_i^{-1}$. We get that $K_i^{-1}$ is compact from a previous lemma. Since a closed subset of a compact set is compact we get that $K_i^{-1} \cap (G \setminus U)$ is compact. Since $K_i^{-1} \cap (G \setminus U)$ is compact it must satisfy the finite intersection property. The next step is to show that the intersection of finitely many $K_i^{-1} \cap (G \setminus U)$ is non-void. To do this we arbitrarily choose two of these sets: $K_n^{-1} \cap (G \setminus U)$ and $K_m^{-1} \cap (G \setminus U)$. It is easy to see that $(K_n^{-1} \cap (G \setminus U)) \cap (K_m^{-1} \cap (G \setminus U))$ $= (K_n^{-1} \cap K_m^{-1}) \cap (G \setminus U)$. Both $K_n^{-1}$ and $K_m^{-1}$ are closed, compact neighbourhoods of $e$, thus $K_n^{-1} \cap K_m^{-1}$ is a closed neighbourhood of $e$. Since $K_n^{-1} \cap K_m^{-1}$ is closed and contained in a compact set, we get that $K_n^{-1} \cap K_m^{-1}$ is compact by Proposition 2.37. Since $K_n^{-1} \cap K_m^{-1}$ is compact it must be equal to $K_i^{-1}$ for some $i \in \mathbb{I}$. Thus $(K_n^{-1} \cap K_m^{-1}) \cap (G \setminus U) \neq \emptyset$. This argument can be extended to any finite intersection of these sets. Next, fix some $K_{i_0}^{-1}$, and say $K_{i_0}^{-1} \cap (G \setminus U) = \mathcal{K}$. Clearly, we have $(K_{i_0}^{-1} \cap K_i^{-1}) \cap (G \setminus U) \subset \mathcal{K}$ for all $i \in \mathbb{I}$. By a similar argument, we see that $[\cap (K_i^{-1} \cap (G \setminus U))] \cap K_{i_0}^{-1} \neq \emptyset$ from the finite intersection property. It follows that $\cap (K_i^{-1} \cap (G \setminus U)) \neq \emptyset$. We also have that $\cap K_i^{-1} = \{e\}$. It follows from $\cap(K_i^{-1} \cap (G \setminus U)) \subset \cap K_i^{-1}$ that $\cap(K_i^{-1} \cap (G \setminus U)) = \{e\}$. This means that $e \in G \setminus U$ and subsequently, $e \notin U$. This is a contradiction since $U$ is a neighbourhood of $e$. It must be that one of the $K_i$'s are contained in $U$. It follows that $G$ is a topological group. $\square$

CHAPTER 4

# Topological Groups

In this chapter we begin the study of topological groups. Here we begin by stating some properties of topological groups and introducing quotient groups. We then give a discussion on compact topological groups and we go over some important theorems and examples of such spaces.

THEOREM 4.1. *Let $G$ be a topological group, the following statements are equivalent:*

(1) *$G$ is a $T_0-$ space*
(2) *$G$ is a $T_1-$ space*
(3) *$G$ is a Hausdorff space*

THEOREM 4.2. *Let $G$ be a topological group. The following maps are homeomorphisms from $G$ to $G$ for all $a \in G$.*

(1) *the translation maps $r_a$ and $l_a$*
(2) *the inverse map: $x \to x^{-1}$*
(3) *the inner automorphism map $x \to axa^{-1}$*

PROOF. The proof for the translation maps is the same as it was for semitopological groups, we proved this in Proposition 3.6. For the inverse mapping: we know it is a bijection since $G$ is a group (every element has a unique inverse), it is continuous by definition of a topological group. The inverse map of the inverse map is itself, so it has a continuous inverse. Thus the inverse map is a topological group. Lastly, we have the automorphism map. This is a composition of two homeomorphisms, $x \to ax$ and $x \to xa^{-1}$, and is thus a homeomorphism. $\square$

COROLLARY 4.3. *Let $G$ be a topological group, let $a \in G$, and let $F$, $P$, $A$ be subsets of $G$ where $F$ is closed, $P$ is open and $A$ is arbitrary. Then, $aF$, $Fa$ and $F^{-1}$ are closed; and $aP$, $Pa$, $P^{-1}$, $AP$, and $PA$ are open.*

PROPOSITION 4.4. *In a topological group $G$ there exists a fundamental system of symmetric neighbourhoods of the identity.*

PROOF. Let $\{V\}$ be a fundamental system of open neighbourhoods of $e$, to complete the proof we show that there is a symmetric subset of $\{V\}$ that satisfies Definition 2.16. Let $V$ be some element of $\{V\}$ with $V$ open, by Theorem 4.2 we get that $V^{-1}$ is an open neighbourhood of $e^{-1}$, but $e^{-1} = e$, so $V^{-1}$ and $V$ are both open neighbourhoods of $e$.

Let $U = V \cap V^{-1}$, $U$ is symmetric since

$$U^{-1} = (V \cap V^{-1})^{-1} = V^{-1} \cap (V^{-1})^{-1} = V^{-1} \cap V = U$$

$U$ is open since it is the intersection of open sets. The set of all such $U$ is a fundamental system of symmetric neighbourhoods of $e$ because for all $V \in \{V\}$ There is some set $U$ with $U \subset V$. $\square$

PROPOSITION 4.5. *In a topological group $G$, let $N$ be a neighbourhood of the identity. Then there exists a symmetric neighbourhood $U$ of $e$ such that $U^n \subset N$ for all integers $n$.*

PROPOSITION 4.6. *Let $G$ be a topological group, let $U$ be a neighbourhood of $e$, and let $H$ be any subset of $G$. We get that $\overline{H} \subset HU$ and $\overline{H} \subset UH$.*

We will now go over some key results for subgroups of topological groups.

PROPOSITION 4.7. *Let $G$ be a topological group. If $H$ is a subgroup of $G$, then $H$ is a topological subgroup of $G$.*

PROPOSITION 4.8. *Let $G$ be a topological group and let $U$ be a symmetric neighbourhood of $e$. We get that $H = \cup_{n\geq 1} U^n$ is a clopen subgroup of $G$.*

PROPOSITION 4.9. *Let $G$ be a topological group.*

(1) *If $H$ is a subgroup of $G$ then so is $\bar{H}$.*

(2) *If $H$ is normal in $G$, then so is $\bar{H}$.*

PROOF. We start by proving part 1. By the subgroup test we need to show:
$\bar{H}$ is nonempty
$x, y \in \bar{H}$ implies $xy \in \bar{H}$ (closure under the operation)
$x \in \bar{H}$ implies $x^{-1} \in \bar{H}$ (closure under inverses)

Since $e \in H$ implies $e \in \bar{H}$, so $\bar{H}$ is nonempty.

Let $h_1, h_2 \in \bar{H}$ and $N \in \mathcal{N}_{h_1 h_2}$. Since we have continuity of the group operation in both variables there exist $V_1 \in \mathcal{N}_{h_1}$ and $V_2 \in \mathcal{N}_{h_2}$ such that $V_1 V_2 \subset N$. From the definition of closure, we get $h_1, h_2 \in \bar{H} = \{x \in G \mid N \cap H \neq \emptyset \ \forall\ N \in \mathcal{N}_x\}$, so there are $x \in V_1 \cap H$ and $y \in V_2 \cap H$ for some $x, y$. This means that $x \in H$, $y \in H$, $x \in V_1$ and $y \in V_2$ . We get that $xy \in V_1 V_2$ and since $H$ is a group we have $xy \in H$. Since $V_1 V_2 \subset N$ we get $xy \in N$, so $xy \in N \cap H$. In order to show $h_1 h_2 \in \bar{H}$ we need to prove $N \cap H \neq \emptyset$ for all $N \in \mathcal{N}_{h_1 h_2}$. We have shown this since $N$ was an arbitrary neighbourhood of $h_1 h_2$ and $xy \in N \cap H$. It follows that $h_1 h_2 \in \bar{H}$, as required.

Let $h \in \bar{H}$ and $N \in \mathcal{N}_{h^{-1}}$, from the continuity of the inverse map we get $N^{-1} \in \mathcal{N}_h$. We have $h \in \bar{H}$, so like before there exists some $x$ with $x \in N^{-1} \cap H$. So $x \in N^{-1}$ and $x \in H$. We have that H is a group so $x^{-1} \in H$ and the continuity of inverses gives us

$x^{-1} \in N$, so $x^{-1} \in N \cap H$. In order to show $h^{-1} \in \bar{H}$ we need to prove $N \cap H \neq \emptyset$ for all $N \in \mathcal{N}_{h^{-1}}$. We have shown this since $N$ was an arbitrary neighbourhood of $h^{-1}$ and $x^{-1} \in N \cap H$. It follows that $h^{-1} \in \bar{H}$, as required.

Thus $\bar{H}$ is a subgroup.

To prove part 2 we need to prove two smaller results first.
i) $\overline{aAa^{-1}} = a\overline{A}a^{-1}$:
Since $\bar{A}$ is closed we have that $a\overline{A}a^{-1}$ is closed by Corollary 4.3. It follows that $\overline{aAa^{-1}} = a\overline{A}a^{-1}$.

ii) $A \subset B$ implies $\bar{A} \subset \bar{B}$
Since $B \subset \bar{B}$ we have that $A \subset \bar{B}$. Since $\bar{B}$ is a closed set that contains $A$, we have $\bar{A} \subset \bar{B}$.

Now to prove the original proposition. We need to show that $g\bar{H}g^{-1} \subset \bar{H}\ \forall g \in G$. By assumption we have $gHg^{-1} \subset H\ \forall g \in G$. From claim ii), we get $\overline{gHg^{-1}} \subset \overline{H}$, from claim i), we get $g\bar{H}g^{-1} \subset \bar{H}$, as desired □

Proposition 4.10. *The centre of a Hausdorff topological group is a closed normal subgroup.*

Proposition 4.11. *The component of the identity of a topological group is a closed normal subgroup.*

Proof. Let $C$ denote the component of $e$. We will first prove that $C$ is closed. Proposition 2.49 states: given a topological space $(X, \tau)$ and $Y$ a dense connected subspace of $X$, then $X$ is connected. For us this means that $C$ is connected implies $\bar{C}$ is connected since $C$ is dense in $\bar{C}$. The component is defined to be the largest connected subset not contained in another connected subset, this forces $C = \bar{C}$ and thus $C$ is closed. Now to prove that $C$ is a subgroup of $G$. Consider the translation $l_{a^{-1}}$ which maps $x \to a^{-1}x$, clearly we have that $l_{a^{-1}}(C) = a^{-1}C$. Since translations are homeomorphisms we know that $a^{-1}C$, is connected. Since $a \in C$ we have $e \in a^{-1}C$, since $a^{-1}C$ is connected and contains $e$ we must have that $a^{-1}C \subset C$. This is true for all $a \in C$ so we have $\cup_{a\in C}\, a^{-1}C = C^{-1}C$. Again, $e \in C^{-1}C$ and $C^{-1}C$ is the union of connected sets and therefore connected by Proposition 2.50, we must have that $C^{-1}C \subset C$. This shows that $C$ is closed under multiplication and inverses and is therefore a group. The final property we need is normality; we need to show that $g^{-1}Cg \subset C\ \forall g \in G$. Let $g$ be an arbitrary element of $G$. Since $e \in C$, we have $g^{-1}eg = g^{-1}g = e \in g^{-1}Cg$. Since the mapping $x \to g^{-1}xg$ is a homeomorphism it follows that $g^{-1}Cg$ is connected. Since $g^{-1}Cg$ is connected and contains the identity we must have $g^{-1}Cg \subset C$, thus $C$ is normal. This completes the proof. □

Theorem 4.12. *Let $G$ be a Hausdorff topological group and let $H$ be a subgroup of $G$. The following propositions hold.*

(1) *$H$ is Hausdorff*
(2) *If $G$ is compact and $H$ is closed, then $H$ is compact*
(3) *If $G$ is locally compact and $H$ is closed, then $H$ is locally compact*

We will now discuss quotient groups.

Definition 4.13. Let $G$ be a topological group and let $H$ be a normal subgroup of $G$. From Definition 2.67 we know that $G/H$ is a group. Let $\phi$ be the mapping from $G$ to $H$ by $\phi(x) = xH$, we will refer to this function as the canonical mapping from $G$ to $G/H$. We can define a topology on $G/H$ as follows: $U$ is open in $G/H$ if and only if $\phi^{-1}(U)$ is open in $G$, we call this the **quotient topology**.

Theorem 4.14. *Let $G/H$ be a topological group with the quotient topology and $\phi$ as above. The following three statements hold:*

(1) *$\phi$ is onto*
(2) *$\phi$ is continuous*
(3) *$\phi$ is open*

Proof. Let $gH \in G/H$. It follows that $\phi$ maps $g$ to $gH$, thus $\phi$ is onto.

Proposition 2.20 tells us that $\phi : G \to G/H$ is continuous when $U$ is open if and only if $\phi^{-1}$ is open. This condition follows directly from the definition of the quotient topology.

Let $U$ be open in $G$. We need to show that $\phi(U)$ is open. We know that $\phi(U) = UH$. By Proposition 3.7 we get that $UH$ is open. □

Theorem 4.15. *Let $G$ be a topological group and let $H$ be normal in $G$. The following statements hold:*

(1) *The canonical mapping $\phi(x) = xH$ is a continuous and open homomorphism.*
(2) *$G/H$ with the quotient topology is a topological group.*

Proof. By Theorem 4.14 we have that $\phi$ is continuous and open. To show that $\phi$ is a homomorphism, let $x, y \in G$. To start, we have $\phi(xy) = xyH$. From the properties of cosets we get $xyH = xHyH$. We can simplify to get $\phi(x)\phi(y)$. Thus $\phi(xy) = \phi(x)\phi(y)$ and $\phi$ is a homomorphism.

To show $G/H$ is a topological group, we need to show that the map defined by $(xH, yH) \to xy^{-1}H$ is continuous. Let $WH$ be an open neighbourhood of $xy^{-1}H$. By the definition of the quotient topology, we know that $\phi^{-1}(WH)$ is open in $G$, $xy^{-1}H \in WH$ implies that $xy^{-1} \in \phi^{-1}(WH)$. By assumption $G$ is a topological group, which means that there exist open sets $U$ and $V$ with $x \in U$ and $y \in V$ such that $UV^{-1} \subset \phi^{-1}(WH)$. Since $xy^{-1} \in UV^{-1}$, it follows that $xy^{-1}H \in \phi(UV^{-1})$; similarly, $UV^{-1} \subset \phi^{-1}(WH)$ implies that $\phi(UV^{-1}) \subset \phi(\phi^{-1}(WH)) = WH$. Combining these two facts we get $xy^{-1} \in \phi(UV^{-1}) \subset WH$. Since $\phi$ is a homomorphism, we get $\phi(UV^{-1}) = \phi(U)\phi(V^{-1})$. It follows that $\phi(U)\phi(V^{-1}) \subset WH$. To complete the proof we need to show that $\phi(U)$ and $\phi(V^{-1})$ are neighbourhoods of $xH$ and $y^{-1}H$, respectively. $xH \in \phi(U)$ and

$y^{-1}H \in \phi(V^{-1})$ follow immediately from the fact that $x \in U$ and $y^{-1} \in V^{-1}$. Since $U$ and $V^{-1}$ are open in $G$, Theorem 4.14 gives us that $\phi(U)$ and $\phi(V^{-1})$ are open, and are thus neighbourhoods. □

EXAMPLE 4.16. *We know that* $(\mathbb{R}^3, \|\cdot\|_3, *)$ *is a topological group. Since* $\mathbb{R}^3$ *is abelian, all subgroups are normal. So* $\mathbb{Q}^2 \times \mathbb{Z}$ *is a normal subgroup of* $\mathbb{R}^3$*. By Theorem 4.15 we get that* $\mathbb{R}^3/\mathbb{Q}^2 \times \mathbb{Z}$ *is a topological group with the quotient topology.*

THEOREM 4.17. *Let* $G$ *be a topological group and let* $H$ *be a closed normal in* $G$*. The following statements hold:*

(1) *If* $G$ *is compact, then* $G/H$ *is compact.*
(2) *If* $G$ *is locally compact, then* $G/H$ *is locally compact.*

PROOF. Assume that $G$ is a compact space. By Proposition 2.36 and the fact that the canonical mapping is continuous and onto we get that $G/H$ is compact.

Assume that $G$ is a locally compact space. This means that there exists some $U \in \mathscr{N}_e$ such that $\bar{U}$ is compact. From Theorem 4.15 we get that $\phi(U)$ is a neighbourhood of $H$ ($H$ is the identity of $G/H$). From Proposition 2.36 we get that $\phi(\overline{U})$ is compact. Notice that $\overline{\phi(U)} \subset \overline{\phi(\overline{U})} = \phi(\overline{U})$. From Proposition 2.37 we get $\overline{\phi(U)}$ is compact since it is a closed subset of a compact set. In $G/H$ there is a neighbourhood of the identity, whose closure is compact. Using translations, we can find such a neighbourhood for an arbitrary point in $G/H$. □

The preceding three theorems also hold for semitopological groups.

We will finish the chapter by giving two theorems on products of topological groups.

THEOREM 4.18. *Let* $\mathbb{I}$ *be an indexed set and let* $G_i$ *be a family of topological groups. Then,* $G = \prod_{i\in\mathbb{I}} G_i$ *is a topological group with the product topology.*

THEOREM 4.19. *Let* $G = \prod_{i\in\mathbb{I}} G_i$ *be a direct product of topological groups, with the product topology. The following statements hold:*

(1) $G$ *is compact if and only if each* $G_i$ *is compact*
(2) $G$ *is Hausdorff if and only if each* $G_i$ *is compact*
(3) *If all but a finite number of* $G_i$ *are compact and all* $G_i$ *are locally compact, then* $G$ *is locally compact.*

PROOF. Assume $G_i$ is compact for all $i \in \mathbb{I}$. From Theorem 2.45 we get that $G$ is compact. For the other direction, assume that $G$ is compact. For all $i \in \mathbb{I}$ we get $\pi_i(G) = G_i$, where $\pi_i$ is the coordinate projection. This mapping is continuous and onto, so $G_i$ is compact by Proposition 2.36.

To prove the second part of the theorem assume that $G_i$ is Hausdorff for all $i \in \mathbb{I}$. Let $x, e \in G$ with $x \neq e$. We have

$$x = (x_1, x_2, .....)$$

$$e = (e_1, e_2, .....)$$

so there must exist some $j \in \mathbb{I}$ such that $x_j \neq e_j$. Since $G_j$ is Hausdorff, there exist some $U_j$ that is an open neighbourhood of $e_j$ in $G_j$ such that $x_j \notin U_j$. Notice that

$$\pi_j^{-1}(U_j) = G_1 \times G_2 \times ... G_{j-1} \times U_J \times G_{j+1} \times ...$$

is an open set in $G$ that contains $e$ and doesn't contain $x$. This means that $G$ is a $T_0-space$. By Theorem 4.1 we get that $G$ is Hausdorff, as desired. For the other direction, assume that $G$ is a Hausdorff space. So for all $x, y \in G \ with \ x \neq y$ there exist open sets $U, V$ with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. For any $j \in \mathbb{I}$ consider the space $G_j$. Let $x_j$ and $y_j$ be two distinct points in $G_j$. Let

$$n = (a_1, a_2, ..., a_{j-1}, x_j, a_{j+1}...)$$
$$m = (a_1, a_2, ..., a_{j-1}, y_j, a_{j+1}...)$$

By assumption, $G$ is Hausdorff, so there exist open sets $N, M$ such that $n \in N$ and $m \in M$ with $N \cap M = \emptyset$. By applying the coordinate projection we get $\pi_j(N) \cap \pi_j(M) = \emptyset$. Thus $G_j$ is Hausdorff. The result follows since $j$ was arbitrary.

Assume that $G_i$ is locally compact for all $i \in \mathbb{I}$ and that there are only finitely many $G_i$ that aren't compact. Suppose that $G_i$ are not compact for $1 \leq i \leq n$ and compact for $1 > n$. By the first part of this theorem, we get get that $\prod_{i>n} G_i$ is compact and therefore locally compact. By Theorem 2.46 we get that $\prod_{1\leq i\leq n} G_i$ is locally compact. So $\prod_{i>n} G_i$ and $\prod_{1\leq i\leq n} G_i$ are two locally compact sets, by applying Theorem 2.46 again we see that $G$ is locally compact. □

CHAPTER 5

# Locally Compact Topological Groups

In this chapter we will focus on locally compact topological groups. Specifically, we will look at groups of matrices. We take an interest in these groups because of their importance in physics. An additional reference for this section is [**6**].

PROPOSITION 5.1. *A topological group is locally compact if and only if there exists a compact neighbourhood of the identity.*

PROOF. Assume that $G$ is a locally compact topological group. By Definition 2.40 there is a neighbourhood $U$ of the identity such that $\overline{U}$ is compact. For the other direction, assume that $U$ is a compact neighbourhood of the identity. By Proposition 4.5 we get that there exists a neighbourhood $V$ of $e$ such that $V^2 \subset U$. Proposition 4.6 tells us that $\overline{V} \subset V^2$. It follows that $\overline{V} \subset \overline{U}$. From Proposition 2.37 we get that $\overline{V}$ is a compact neighbourhood of $e$. Given any $x \in G$ we have that $xV$ is a neighbourhood of $x$. Since a translation by $x$ is a homeomorphism we get that $\overline{xV}$ is a compact neighbourhood of $x$. It follows that $G$ is locally compact. $\square$

THEOREM 5.2. *Let $G$ be a topological group and let $U$ be a compact open neighbourhood of the identity. Then $U$ contains a compact clopen subgroup of $G$.*

THEOREM 5.3. *Let $G$ be a compact topological group and let $U$ be a clopen neighbourhood $e$. Then $U$ contains a clopen, normal subgroup $M$ of $G$.*

PROOF. Since $G$ is compact and $U$ is closed, we get that $U$ is compact form Proposition 2.37. By Theorem 5.3 we get there exists a clopen compact subgroup $H$ of $G$ that is contained in $U$. Let $M = \cap_{x \in G} xHx^{-1}$ $\square$

Now we will discuss groups of matrices. These are an example of locally compact topological groups.

## 1. Groups of Matrices

Let $\mathbb{F}$ be a field, for the cases that we will be investigating we will either have $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Recall that $\mathbb{F}^n$ denotes the direct product of $\mathbb{F}$ with itself $n$-times. Elements in $\mathbb{F}^n$ look like $x = (x_1, x_2, ..., x_n)$, where $x_i \in \mathbb{F}$.

DEFINITION 5.4. *A function $f : \mathbb{F}^n \to \mathbb{F}^n$ is called an* **endomorphism** *if the following holds.*

(1) *$f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{F}^n$*
(2) *$f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$*

Definition 5.5. Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of order $m \times n$. The sum of $A$ and $B$, denoted $A + B$ is defined to be the $m \times n$ matrix with entries $c_{ij}$ where $c_{ij} = a_{ij} + b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

For the sum to be defined the two matrices being added must be of the same size. Matrix addition is communative since addition in $\mathbb{F}$ is communative.

Definition 5.6. *Let $A = (a_{ij})$ be a $m \times n$ matrix and let $B = (b_{ij})$ be a $n \times l$ matrix. The product $AB$ is defined to be the $m \times l$ matrix given by $(d_{ij})$ where $d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.*

In order for the product $AB$ to be defined the number of rows of $A$ must be the same as the number of columns in $B$. In general, matrix multipliaction is not communative.

Proposition 5.7. *For each endomorphism of $\mathbb{F}^n$ there corresponds a matrix of order $n \times n$. For each matrix of order $n \times n$ there corresponds an endomorphims of $\mathbb{F}^n$.*

Proof. Recall that $e_i = (0, 0, .., 0, 1, 0, ..., 0)$ is the vector in $\mathbb{F}^n$ with 1 in the $i^{th}$ component and zeroes elsewhere. Let $f$ be an endomorphism of $\mathbb{F}^n$. We can write $f$ as $f(e_i) = \sum_{j=1}^{n} a_{ij} e_j = (a_{i1}, a_{i2}, \ldots, a_{in})$. We get that $A = (a_{ij})$ is an $n \times n$ matrix. We have that for a function $f$ there corresponds a matrix $A$. We will now show that $f(x)$ is equivalent to $xA$. Let $x \in \mathbb{F}^n$. We can write $x$ as $x = (\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i=1}^{n} \lambda_i e_i$. We get that:

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{n} f(\lambda_i e_i) \\
&= \sum_{i=1}^{n} \lambda_i f(e_i) \quad (\textit{since } f \textit{ is linear}) \\
&= \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} a_{ij} e_i \quad (\textit{by construction of } f) \\
&= \sum_{j=1}^{n} (\sum_{i=1}^{n} \lambda_i a_{ij}) e_i \quad (\textit{by rearranging terms}) \\
&= \sum_{i=1}^{n} \lambda_i a_{i1} e_1 \, + \sum_{i=1}^{n} \lambda_i a_{i2} e_2 \, + \, ... \, + \sum_{i=1}^{n} \lambda_i a_{in} e_n \quad (\textit{expanding the summation}) \\
&= (\sum_{i=1}^{n} \lambda_i a_{i1}, \, \sum_{i=1}^{n} \lambda_i a_{i2}, \, ..., \sum_{i=1}^{n} \lambda_i a_{in}) \quad (\textit{seperating the components}) \\
&= (\lambda_1, \lambda_2, ..., \lambda_n) A \quad (\textit{factoring out } \lambda_i) \\
&= xA
\end{aligned}
$$

Conversely, assume that $A = (a_{ij})$ is an $n \times n$ matrix. For any $x \in \mathbb{F}^n$, $xA$ is a vector in $\mathbb{F}^n$. It is a well known property of matrices that $(x+y)A = xA+yA$ and $\lambda xA = \lambda(xA)$. From this we get an endomorphism. This completes the proof.

□

The definition and proposition that follow are a review of basic linear algebra.

DEFINITION 5.8. *Let $A = (a_{ij})$ be a matrix of order $n \times n$.*

(1) *The* **transpose** *of $A$, denoted by $A^T$, is the matrix obtained whose columns are the rows of $A$. In symbols: $A^T = (a_{ji})$.*
(2) *The* **complex conjugate** *of $A$, denoted by $\overline{A}$, is the matrix obtained by taking the complex conjugate of each entry of $A$. In symbols: $\overline{A} = (\overline{a_{ij}})$.*
(3) *The matrix $A^\dagger$ is called the* **Hermitian conjugate** *of $A$. The matrix $A^\dagger$ is given by $A^\dagger = (\overline{A})^T$*
(4) *The matrix $A$ is* **invertable** *means that there exists an $n \times n$ matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$, where $I$ is the identity matrix.*
(5) *The* **determinant** *of $A$, denoted $det(A)$, is defined to be*

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

*Here $sgn(\sigma)$ denotes the signature of the permutation $\sigma$. By definition $sgn(\sigma) = 1$ if $\sigma$ is even, and $-1$ if $\sigma$ is odd.*

PROPOSITION 5.9. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of order $n \times n$.*

(1) *$A$ is a matrix of real entries if and only if $A = \overline{A}$.*
(2) $(AB)^T = B^T A^T$
(3) $\overline{AB} = \overline{A}\,\overline{B}$
(4) *$A$ is invertible if and only if $det(A) \neq 0$*
(5) *If $A$ and $B$ are invertible, then $AB$ and $BA$ are invertible and $(AB)^{-1} = B^{-1}A^{-1}$*

Next, we will give an example of a topology on $M_n(\mathbb{F})$.

Let $A = (a_{ij})$ be an element of $M_n(\mathbb{F})$. Since $A$ is an $n \times n$ matrix, it has $n^2$ entries. We can think of $A$ as a point in $\mathbb{F}^{n^2}$. This is because we can take the $n^2$ entries in $A$ and put them in some order. Let $f$ be the function that associates that ordered list to an element in $\mathbb{F}^{n^2}$. It is clear that $f$ is a bijection. We choose a topology on $\mathbb{F}^{n^2}$ and equip $M_n(\mathbb{F})$ with that same topology. So, a set $U \in M_n(\mathbb{F})$ is open if and only if $f(U)$ is open in $\mathbb{F}^{n^2}$.

EXAMPLE 5.10. *Consider the topological space $(\mathbb{R}^{n^2}, \|\cdot\|_n)$. Then $M_n(\mathbb{R})$ is a topological space.*

PROPOSITION 5.11. *$M_n(\mathbb{F})$ with the Euclidean topology is a topological group that is Hausdorff, noncompact, and locally compact.*

Proof. The result follows immediately from the fact that $\mathbb{F}^{n^2}$ is a Hausdorff, non-compact, and locally compact topological group with the Euclidean topology. $\square$

Proposition 5.12. *$GL_n(\mathbb{F})$ is an open subset of $M_n(\mathbb{F})$.*

Proof. Let $A$ be an element of $M_n(\mathbb{F})$ and let $f(A) = det(A)$ be a function from $M_n(\mathbb{F})$ to $F$. Since $GL_n(\mathbb{F})$ contains the matrices of $M_n(\mathbb{F})$ with non-zero determinant we get $GL_n(\mathbb{F}) = M_n(\mathbb{F}) \setminus f^{-1}(\{0\})$. We know that $f$ is continuous because the determinant function is a polynomial and polynomials are continuous. Since $\{0\}$ is closed in $\mathbb{R}$ we get that $f^{-1}(\{0\})$ is closed in $M_n(\mathbb{F})$. It follows that $GL_n(\mathbb{F})$ is open. $\square$

Proposition 5.13. *$GL_n(\mathbb{F})$ is a Hausdorff topological group when equipped with the relative topology induced from $M_n(\mathbb{F})$.*

Theorem 5.14. *Let $A^* = (A^T)^{-1}$. The following maps are homeomorphims from $GL_n(\mathbb{F})$ to $GL_n(\mathbb{F})$:*

(1) $A \to A^{-1}$
(2) $A \to \overline{A}$
(3) $A \to A^T$
(4) $A \to A^*$

Definition 5.15. *Let $A$ be a matrix in $GL_n(\mathbb{F})$. The following are three special subgroups of $GL_n(\mathbb{F})$.*

(1) *We call $A$* **orthogonal** *when $A = \overline{A} = A^*$. The set of all orthogonal matrices in $GL_n(\mathbb{F})$ is called the orthogonal group, denoted $O_n(\mathbb{R})$.*
(2) *We call $A$* **complex orthogonal** *when $A = A^*$. The set of all complex orthogonal matrices in $GL_n(\mathbb{F})$ is called the complex orthogonal group, denoted $O_n(\mathbb{C})$.*
(3) *We call $A$* **unitary** *when $\overline{A} = A^*$. The set of all unitary matrices in $GL_n(\mathbb{F})$ is called the unitary group, denoted $U_n$.*

By the construction of these groups it should be obvious that $O_n(\mathbb{R}) = O_n(\mathbb{C}) \cap U_n$ and $O_n(\mathbb{R}) = GL_n(\mathbb{R}) \cap O_n(\mathbb{C})$.

Definition 5.16. There are three more special subgroups that we now define

(1) The **special full linear group** $SG_n(\mathbb{C})$ is the subgroup of $GL_n(\mathbb{C})$ that contains only the matrices with determinant 1.
(2) The **special orthogonal group** $SO_n(\mathbb{R})$ is the subgroup of $O_n(\mathbb{R})$ that contains only the matrices with determinant 1.
(3) The **special unitary group** $SU_n$ is the subgroup of $U_n$ that contains only the matrices with determinant 1.

Proposition 5.17. *The following statements are equivalent*

(1) *$A$ is unitary*
(2) $A^T\,\overline{A} = \overline{A}A^T = I$

(3) $A^{\dagger} = A^{-1}$
(4) $\overline{A}^{-1} = A^T$

PROPOSITION 5.18. *$O_n(\mathbb{R})$, $O_n(\mathbb{C})$, and $U_n$ are closed subgroups of $GL_n(\mathbb{C})$.*

THEOREM 5.19. *$O_n(\mathbb{R})$ and $U_n$ are compact topological subgroups of $GL_n(\mathbb{C})$.*

PROOF. First we will show that $U_n$ is compact. Let $A \in U_n$, it follows that $\overline{A} = A^* = (A^T)^{-1}$, which implies that $A^T\,\overline{A} = I$. Recall that $I = (\delta_{ij})$, where $\delta_{ij}$ is the Kronecker delta, which is defined to be

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

By the definition of matrix multiplication we get that

$$A^T\,\overline{A} = \sum_{k=1}^{n} a_{ki}\overline{a_{kj}} = \delta_{ij}$$

For $i = j$ we get

$$\sum_{k=1}^{n} a_{ki}\overline{a_{ki}} = \sum_{k=1}^{n} |a_{ki}|^2 = 1$$

where $|a_{ki}|^2$ is the modulus squared of $a_{ki}$. It follows that $|a_{ki}| \leq 1$ for all $1 \leq i, k \leq n$. Consider the identity mapping $f : M_n(\mathbb{C}) \to \mathbb{C}^{n^2}$, we know that $\mathbb{C}^{n^2}$ is isomorphic to $\mathbb{R}^{2n^2}$, so let $f : M_n(\mathbb{C}) \to \mathbb{R}^{2n^2}$. We have that $f$ maps $U_n$ into $[0, 1]^{2n^2} \subset \mathbb{R}^{2n^2}$. Since $[0, 1]^{2n^2}$ is bounded, it follows that $f(U_n)$ is bounded. By Proposition 5.18 we get that $U_n$ is closed. $f$ is a homeomorphism thus $f(U_n)$ is closed in $\mathbb{R}^{2n^2}$. From the Heine-Borel theorem we get that $f(U_n)$ is compact. Since $f$ is a homeomorphism, $U_n$ is compact in $GL_n(\mathbb{C})$. From Proposition 5.13 we get that $U_n$ is compact in $GL_n(\mathbb{C})$. If follows from Proposition 2.37 that $O_n(\mathbb{R})$ is compact in $GL_n(\mathbb{C})$. $\square$

COROLLARY 5.20. *$SO_n(\mathbb{R})$ and $SU_n$ are compact subgroups of $GL_n(\mathbb{C})$.*

PROOF. Let $f : M_n(\mathbb{C}) \to \mathbb{C}$ be a function given by $A \to det(A)$. As discussed earlier this function is continuous. Both $SO_n(\mathbb{R})$ and $SU_n$ are closed subsets of $M_n(\mathbb{C})$ since they are the inverse image of $\{1\}$, and $\{1\}$ is a closed set in $\mathbb{C}$. So $SO_n(\mathbb{R})$ and $SU_n$ are closed and are contained in the compact sets $O_n(\mathbb{R})$ and $U_n$, respectively. The result follows from Theorem 5.19. $\square$

EXAMPLE 5.21. *The groups $SO_n(\mathbb{R})$ and $SU_n$ are of great interest to physicists. The groups $SU_2$ and $SU_3$ are used to describe systems with rotational symmetries.*

*In quantum mechanics, the Pauli matrices are elements of $U_2$ that are used to describe the spin of spin-$\frac{1}{2}$ particles, like electrons, in three spacial dimensions.*

*These matrices are given by*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*The set $\{i\sigma_x, i\sigma_y, i\sigma_x, e\}$ where $e$ is the identity matrix is a basis for $SU_2$.*

CHAPTER 6

# Concluding Remarks

Up to this point we have have given a basic introduction to topological groups. We have given a review of topological spaces and group theory. With respect to topological groups, we have gone over many key results in the area. Topological quotient groups and groups of matrices have been discussed. From here the next topic to study is the Haar measure. When we have a locally compact topological group, we can use the Haar measure. In order to give an introduction to the Haar measure we first need to define a few terms. Our references for this section are [**7**] and [**2**].

DEFINITION 6.1. *A collection $\Sigma$ of subsets of a set $X$ is a $\sigma -$ **algebra** when the following hold:*

(1) $\emptyset \in \Sigma$ *and* $X \in \Sigma$
(2) *If* $A \in \Sigma$*, then* $X \backslash A \in \Sigma$
(3) *If* $(A_n)$ *is a sequence of sets in* $\Sigma$*, then the union* $\cup_{n=1}^{\infty} A_n \in \Sigma$

The pair $(X, \Sigma)$ is called a measurable space.

DEFINITION 6.2. *Let $X$ be a set and let $\Sigma$ be a $\sigma -$ algebra over $X$. A function $\mu : \Sigma \to [0, \infty)$ is a* **measure** *when it satisfies the following properties:*

(1) $\mu(E) \geq 0$ *for all* $E \in \Sigma$
(2) $\mu(\emptyset) = 0$
(3) $\mu(\cup_{i \in \mathbb{I}} E_i) = \sum_{i \in \mathbb{I}} \mu(E_i)$ *for all countable collections of pairwise disjoint* $E_i$ *in* $\Sigma$

DEFINITION 6.3. *Given a topological group $(G, \tau, *)$, we say that $\Sigma$ is a* **Borel algebra** *when $\Sigma$ is generated by all open sets in $\tau$.*

An element of the Borel algebra is called a Borel set.

DEFINITION 6.4. *A* **Haar measure** *on a a locally compact topological group $(G, \tau, *)$ is a measure $\mu : \Sigma \to [0, \infty)$, where $\Sigma$ is a $\sigma -$* **algebra** *that contains all Borel sets of $(G, \tau, *)$ , such that $\mu(gS) = \mu(S)$ for all $g \in G$ and $S \in \Sigma$*

With this we can talk about integration on locally compact topological groups.

THEOREM 6.5. *Let $G$ be a locally compact topological group. There exists a Haar measure $\mu$ on $G$*

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