Linear Functional Analysis

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Abstract

This project is designed to give readers a basic understanding on the topic of Linear Functional Anaylsis. After covering preliminaries, we will examine different spaces and their properties, bounded linear operators, duality, and finish with the Hahn-Banach Theorem - a very powerful theorem that is a cornerstone of functional analysis and has applications in other areas of mathematics as well.

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Contents

CHAPTER 1

Introduction

This project will introduce the methods of linear functional analysis. Our basic goal here is to perform analysis on infinite-dimension vector spaces. Because not all of our linear algebra properties hold for infinite spaces, we extend ideas like the length of a vector into an abstract setting: the norm. Much like in real and complex analysis, we will give additional structures to certain spaces to yield special properties. Through different examples, we will gain an understanding of normed and inner product spaces, bounded linear operators, duality, and the Hahn-Banach theorem.

CHAPTER 2

Preliminaries

1. Linear Algebra

Here, we will recall some essential concepts from Linear Algebra for use throughout the project. We will be working with the sets $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$. Note that $\mathbb{R}$ and $\mathbb{C}$ are both algebraic *fields*, and when giving examples, theorems, or definitions that apply equally to $\mathbb{R}$ and $\mathbb{C}$, we will simply use $\mathbb{F}$ to denote either set.

DEFINITION 2.1. A *vector space* over $\mathbb{F}$ is a non-empty set V with two operations, *vector addition* and *scalar multiplication*, where vector addition is a function from $V \times V \to V$ defined by $(x, y) = x + y$, and scalar multiplication is a function from $\mathbb{F} \times V \to V$ defined by $(\alpha, x) = \alpha x$. The following properties hold for any $\alpha, \beta \in \mathbb{F}$ and any $x, y, z \in V$.

(i) $x + y = y + x, \quad x + (y + z) = (x + y) + z$;
(ii) there exists a unique $0 \in V$ such that for all $x \in V$, $x + 0 = 0 + x = x$;
(iii) for all $x \in V$, there exists a unique $-x \in V$ such that $x + (-x) = (-x) + x = 0$;
(iv) $1x = x, \quad \alpha(\beta x) = (\alpha\beta)x$;
(v) $\alpha(x + y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta x$.

Note that if $\mathbb{F} = \mathbb{R}$, then V is a *real* vector space, and if $\mathbb{F} = \mathbb{C}$, then V is a *complex* vector space. For our purposes, we will usually just use the term "vector space," as most results about vector spaces will apply equally well to both the real and complex case.

DEFINITION 2.2. Let V be a vector space, and U a non-empty subset of V. Then U is a *linear subspace* of V if U is a vector space under the same vector addition and scalar multiplication as V.

Rather than proving Definition 2.1 (i)-(v) for U, we can use the *subspace test*:
Let U be a non-empty subset of a vector space V. For all $\alpha \in \mathbb{F}$ and $x, y \in U$, if

(i) $x + y \in U$;
(ii) $\alpha x \in U$

then U is a subspace of V. Note that the set $\{\mathbf{0}\} \subset V$ is a linear subspace.

DEFINITION 2.3. Let V be a vector space, $\mathbf{v} = \{v_1, v_2, \ldots, v_k\} \subset V, k \geq 1$ be a finite set, and A be an arbitrary non-empty subset of V.

(i) A *linear combination* of $\mathbf{v}$ is any vector x of the form:

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \in V,$$

for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$.

(ii) The set $\mathbf{v}$ is *linearly independent* if:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

(iii) The *span* of A (denoted SpA) is the set of all linear combinations of all finite subsets of A. In other words, SpA is the intersection of all linear subspaces of V containing A. By this definition, SpA is the smallest linear subspace of V that still contains A.

(iv) Let $\mathbf{v}$ be linearly independent, and Sp$\mathbf{v} = V$. Then $\mathbf{v}$ is a *basis* of V. When V has such a finite basis with k elements, every other basis of V will also have k elements. We say that V is *k-dimensional* and write dim$V = k$.

(v) The set $\mathbb{F}^k$ is a vector space over $\mathbb{F}$, and the set of vectors

$$\hat{e}_1 = (1, 0, 0, \ldots, 0), \quad \hat{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \hat{e}_k = (0, 0, 0, \ldots, 1)$$

forms a basis for $\mathbb{F}^k$, known as the *standard basis*.

Definition 2.4. Let V and W be vector spaces over $\mathbb{F}$. Then the *Cartesian product* $V \times W$ is a vector space, where vector addition and scalar multiplication are defined as follows: For any $\alpha \in \mathbb{F}$ and $(x_1, y_1), (x_2, y_2) \in V \times W$,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \;\; \textit{and} \;\; \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Definition 2.5. Let V and W be vector spaces over $\mathbb{F}$. Then a function $T : V \to W$ is a *linear transformation* if, for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in V$,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

We define a set of all linear transformations $T : V \to W$, denoted by $L(V, W)$, which is also a vector space. If $V = W$, then we will shorten $L(V, V)$ to just $L(V)$. A notable element of $L(V)$ is the *identity transformation* on V, defined by $I_V(x) = x$ for $x \in V$.

2. Metric Spaces

Metric spaces are an abstract setting for the discussion of concepts from analysis (such as convergence and continuity). The basic tool used here is the notion of a distance function, or a metric, which we will define in this section.

Definition 2.6. A *metric* on a set M is a function $d : M \times M \to \mathbb{R}$ such that for all $x, y, z \in M$:

(i) $d(x, y) \geq 0$;

(ii) $d(x, y) = 0 \Leftrightarrow x = y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (the *triangle inequality*).

If d is a metric on M, then we called the pair (M, d) a *metric space*. Unless the set M consists of a single point, it can have more than one metric, and we often say "the metric space M" rather than (M, d).

EXAMPLE 2.7. For an integer $k \geq 1$, the function d : $\mathbb{F}^k \times \mathbb{F}^k \to \mathbb{R}$ defined by

$$d(x, y) = \left(\sum_{j=1}^{k} |x_j - y_j|^2 \right)^{1/2},$$

is the *standard metric* on the set $\mathbb{F}^k$. Thus we call $\mathbb{F}^k$ a metric space.

DEFINITION 2.8. A sequence $\{x_n\}$ in a metric space M is *convergent* if for some $x \in M$ and every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon \text{ for all } n \geq N.$$

If this is true, we write $\lim_{n\to\infty} x_n = x$, or $x_n \to x$.

DEFINITION 2.9. A sequence $\{x_n\}$ in a metric space M is a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon \text{ for all } m, n \geq N.$$

DEFINITION 2.10. Let (M, d) be a metric space and $A \subset M$.

(i) A is *bounded* if there exists a number $B > 0$ such that $d(x, y) < B$ for all $x, y \in A$.
(ii) A is *open* if for all $x \in A$, there exists an $\epsilon > 0$ such that $B_x(\epsilon) = \{y \in M : d(x, y) < \epsilon\} \subset A$. We call $B_x(\epsilon)$ the *open ball* with radius ϵ centered at x.
(iii) A is *closed* if the set $M \setminus A$ is open.
(iv) A point $x \in M$ is a *closure point* of A if for every $\epsilon > 0$, there exists a $y \in A$ with $d(x, y) < \epsilon$ (we can also say, if there exists an $\{y_n\} \subset A$ such that $y_n \to x$).
(v) The *closure* of A, denoted $\bar{A}$, is the set of all closure points in A.
(vi) A is *dense* in M if $\bar{A} = M$.

Note that if $x \in A$, then x is a closure point of A (for every $\epsilon > 0$, $d(x, x) = 0 < \epsilon$), hence $A \subset \bar{A}$. However, the converse is not necessarily true, as we see in the following theorem.

THEOREM 2.11. *Let (M, d) be a metric space and $A \subset M$.*

(i) *$\bar{A}$ is closed and is the smallest closed set that contains A.*
(ii) *A is closed, if and only if $A = \bar{A}$.*
(iii) *A is closed if and only if the following implication holds:*
if $\{x_n\}$ is a sequence in A and $x \in M$ such that $x_n \to x$, then $x \in A$.
(iv) *$x \in \bar{A}$ if and only if inf$\{d(x, y) : y \in A\} = 0$.*
(v) *A is dense if and only if for any element $x \in M$, there exists a sequence $\{y_n\}$ in A such that $y_n \to x$.*

We can extend the idea of a continuous function from real analysis to describe general metric spaces.

DEFINITION 2.12. Let (M, d_M) and (N, d_N) be metric spaces, and $f : M \to N$.

(i) f is *continuous at a point* $x \in M$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $y \in M$,

$$d_M(x, y) < \delta \Rightarrow d_N(f(x), f(y)) < \epsilon.$$

(ii) f is *continuous* on M if it is continuous at each point of M.

(iii) f is *uniformly continuous* on M if for all $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x, y \in M$,

$$d_M(x, y) < \delta \Rightarrow d_N(f(x), f(y)) < \epsilon.$$

DEFINITION 2.13. A metric space (M, d) is *complete* if every Cauchy sequence in (M, d) is convergent. Furthermore, a set $A \subset M$ is *complete in* (M, d) if every Cauchy sequence in A converges to an element of A.

EXAMPLE 2.14. For $k \geq 1$, the space $\mathbb{F}^k$ with the standard metric is complete.

DEFINITION 2.15. Let (M, d) be a metric space. A set $A \subset M$ is *compact* if every sequence in A contains a subsequence converging to an element of A. A set $A \subset M$ is *relatively compact* if $\bar{A}$ is compact.

THEOREM 2.16. *Let (M, d) be a metric space and $A \subset M$. Then:*

(i) *if A is complete, then A is closed;*

(ii) *if M is complete, then A is complete if and only if A is closed;*

(iii) *if A is compact, then A is closed and bounded;*

(iv) *every closed and bounded subset of $\mathbb{F}^k$ is compact (Bolzano-Weirstrass theorem).*

THEOREM 2.17. *Let (M, d) be a compact metric space and $f : M \to \mathbb{F}$ be continuous. Then f is bounded. Particularly, if $\mathbb{F} = \mathbb{R}$, then the supremum and infimum of $f(x)$ exist and are finite, and f attains the values $\inf\{f(x) : x \in M\}$ and $\sup\{f(x) : x \in M\}$.*

DEFINITION 2.18. Let (M, d) be a compact metric space. Then $C_{\mathbb{F}}(M)$ denotes the set of continuous functions $f : M \to \mathbb{F}$. We will define the *uniform metric* on $C_{\mathbb{F}}(M)$ by

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in M\}.$$

Note that most properties of $C_{\mathbb{F}}(M)$ hold for both real and complex cases, so generally, we will omit the subscript and just write $C(M)$. Furthermore, if M is a bounded interval $[a, b] \subset \mathbb{R}$, we will write $C[a, b]$. Also note that $(C(M), d)$ is a complete metric space.

THEOREM 2.19. *The metric space C(M) is complete.*

3. Lebesgue Integration

So far we have seen only the uniform metric on $C[a,b]$, but there are other metrics that are defined by integrals. If $\int_a^b f(x)dx$ is the usual Riemann integral of $f \in C[a,b]$, then for $1 \leq p < \infty$, the function $d_p : C[a,b] \times C[a,b] \to \mathbb{R}$ defined by

$$d_p(f,g) = \left(\int_a^b |f(x) - g(x)|^p \; dx \right)^{1/p}$$

is a metric on $C[a,b]$. However, $(C[a,b], d_p)$ is not a complete metric space, due to some drawbacks of the Riemann integral. Thus, we will introduce here the Lebesgue Integral, which is more powerful and works on a wider class of functions. We will look at metric spaces which are complete using the metric d_p, and will introduce the notion of a "measure" of a set (AKA the "length;" for example, the bounded interval $[a,b]$ has length $b-a$). Note that many sets such as $\mathbb{R}$ have "infinite measure," and so we introduce the extended sets of real numbers: $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $\overline{\mathbb{R}}^+ = [0,\infty) \cup \{\infty\}$. The standard algebraic operations apply, with the products $0 \cdot \infty = 0 \cdot (-\infty) = 0$, and the operations $\infty - \infty$ and ∞/∞ forbidden (like dividing by zero in $\mathbb{R}$).

Definition 2.20. A *σ-algebra* is a collection, Σ, of subsets of X satisfying:

(i) $\emptyset, X \in \Sigma$;
(ii) $S \in \Sigma \Rightarrow X \setminus S \in \Sigma$;
(iii) $S_n \in \Sigma, n = 1, 2, \ldots, \Rightarrow \bigcup_{n=1}^{\infty} S_n \in \Sigma$.

A set $S \in \Sigma$ is said to be *measurable*.

Definition 2.21. Let X be a set and Σ be a *σ-algebra* of subsets of X. A function $\mu : \Sigma \to \overline{\mathbb{R}}^+$ is a *measure* if it has the following properties:

(i) $\mu(\emptyset) = 0$;
(ii) μ is *countably additive*. In other words, if $S_j \in \Sigma, j = 1, 2, \ldots,$ are pairwise disjoint sets, then

$$\mu\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} \mu(S_j).$$

The triple (X, Σ, μ) is called a *measure space*.

Definition 2.22. Let (X, Σ, μ) be a measure space. Then a set $S \in \Sigma$ with $\mu(S) = 0$ is said to have *measure zero*. Also, a given property $P(x)$ is said to hold *almost everywhere* if the set $\{x : P(x) \text{ is false}\}$ has measure zero.

Definition 2.23. Let $X = \mathbb{N}$, Σ_c be the class of all subsets of $\mathbb{N}$, and for any $S \subset \mathbb{N}$, define $\mu_c(S)$ to be the number of elements of S. Then Σ_c is a σ-algebra and μ_c is a measure on Σ_c called the *counting measure* on $\mathbb{N}$. Note that the only set of measure zero in $(\mathbb{N}, \Sigma_c, \mu_c)$ is the empty set.

Definition 2.24. Let $X = \mathbb{R}$, Σ_L be a *σ-algebra*, and μ_L be a measure on Σ_L, such that any finite interval $I = [a, b] \in \Sigma_L$ and $\mu_L(I) = \ell(I)$. Then Σ_L is called the *Lebesgue measure* and the sets in Σ_L are *Lebesgue measurable.*

Note that the sets of measure zero in $(\mathbb{R}, \Sigma_L, \mu_L)$ are the sets A with the following property: for any $\epsilon > 0$, there exists a sequence of intervals $I_j \subset \mathbb{R}, j = 1, 2, \ldots,$ such that

$$A \subset \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} \ell(I_j) < \epsilon.$$

Definition 2.25. A function $\phi : X \to \mathbb{R}$ is *simple* if it has the form

$$\phi = \sum_{j=1}^{k} \alpha_j \chi_{S_j},$$

for some $k \in \mathbb{N}$ where $\alpha_j \in \mathbb{R}$ and $S_j \in \Sigma, j = 1, \ldots, k$. If ϕ is non-negative and simple, then the *integral* of ϕ over X with respect to μ is defined by

$$\int_X \phi d\mu = \sum_{j=1}^{k} \alpha_j \mu(S_j).$$

We do allow $\mu(S_j) = \infty$ here, and use the algebraic rules of $\overline{\mathbb{R}}^+$ from the beginning of this section to evaluate (since ϕ is non-negative, we will not encounter any problems like $\infty - \infty$). The value of the integral may be ∞.

Definition 2.26. Let (X, Σ, μ) be a measure space. Then a function $f : X \to \overline{\mathbb{R}}$ is *measurable* if, for every $\alpha \in \mathbb{R}$,

$$\{x \in X : f(x) > \alpha\} \in \Sigma.$$

Definition 2.27. If f is non-negative and measurable then the *integral* of f is defined to be

$$\int_X f d\mu = \sup \left\{ \int_X \phi d\mu : \phi \text{ is simple and } 0 \leq \phi \leq f \right\}.$$

Definition 2.28. If f is measurable and $\int_X |f| d\mu < \infty$, then f is *integrable*, and the *integral* of f is defined to be

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu,$$

where $f^{\pm}(x) = \max\{\pm f(x), 0\}$. The set of $\mathbb{F}$-valued integrable functions on X will be denoted by $\mathcal{L}^1(X)$

Now we would like to define a metric on $\mathcal{L}^1(X)$, and the obvious choice is

$$d_1(f, g) = \int_X |f - g| d\mu,$$

for all $f, g \in \mathcal{L}^1(X)$. However, this doesn't fit criteria (ii) of Definition 2.6 for being a metric. Unfortunately, $\mathcal{L}^1(X)$ has functions f, g such that $f = g$ almost everywhere, but

$f \neq g$ (which may happen on a set of measure zero). Hence we may have $d_1(f,g) = 0$, but $f \neq g$. So, we will regard "equivalent" any two functions $f, g \in \mathcal{L}^1(X)$ which are equal almost everywhere. We will define an equivalence relation $\equiv$ on $\mathcal{L}^1(X)$ by

$$f \equiv g \iff f(x) = g(x) \text{ almost everywhere for all } x \in X.$$

This partitions $\mathcal{L}^1(X)$ into a space of equivalence classes, which we will denote by $L^1(X)$. The space $L^1(X)$ is a vector space and we see that $d_1(f,g) = 0$ if and only if $f \equiv g$. Thus d_1 meets all criteria to be a metric for $L^1(X)$, which we will use from now on instead of $\mathcal{L}^1(X)$. We should also note that when we say the "function" $f \in L^1(X)$, we mean some representative of the appropriate equivalence class.

Example 2.29. Take the counting measure $(\mathbb{N}, \Sigma_c, \mu_c)$. Any function $f : \mathbb{N} \to \mathbb{F}$ can be seen as an $\mathbb{F}$-valued sequence $\{a_n\}$ (with $a_n = f(n)$ for $n \geq 1$) such that every sequence is a measurable function. It follows, then, that $\{a_n\}$ is integrable with respect to μ_c if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$. The space of these sequences will be denoted by ℓ^1.

Definition 2.30. Let $(X, \Sigma, \mu) = (\mathbb{R}^k, \Sigma_L, \mu_L)$ for some $k \geq 1$. If $f \in \mathcal{L}^1(X)$, then f is *Lebesgue integrable.*

Theorem 2.31. *If $I = [a,b]$ is a bounded interval and $f : I \to \mathbb{R}$ is bounded and Riemann integrable on I, then f is Lebesgue integrable on I, and the values of the two integrals are the same. Particularly, continuous functions on I are Lebesgue integrable, and we use the same notation as the Riemann integral.*

Definition 2.32. We define the space $\mathcal{L}^p(X)$ as follows:

$$\mathcal{L}^p(X) = \{f : f \text{ is measurable and } (\int_X |f|^p d\mu)^{1/p} < \infty\} \text{ when } 1 \leq p < \infty.$$

We also define the corresponding sets $L^p(X)$ by identifying functions in $\mathcal{L}^p(X)$ which are equal almost everywhere, and considering corresponding spaces of equivalence classes.

Example 2.33. For $1 \leq p < \infty$, the function $d_p : L^p(X) \times L^p(X) \to \mathbb{R}$ defined by

$$d_p(f,g) = \left(\int_X |f - g|^p d\mu \right)^{1/p}$$

is the *standard metric* on $L^p(X)$.

Example 2.34 (Counting Measure). Let $1 \leq p \leq \infty$ and $(X, \Sigma, \mu) = (\mathbb{N}, \Sigma_c, \mu_c)$. Then the space $L^p(\mathbb{N})$ consists of the set of sequences $\{\alpha_n\}$ in $\mathbb{F}$ with the properties:

$$\left(\sum_{n=1}^{\infty} |\alpha_n|^p \right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

$$\sup \{|\alpha_n| : n \in \mathbb{N}\} < \infty \quad \text{for } p = \infty.$$

These measure spaces will be denoted ℓ^p.

Theorem 2.35. *Let $1 \leq p \leq \infty$. Then the metric space $L^p(X)$ is complete, and in particular, the sequence space ℓ^p is complete.*

CHAPTER 3

Spaces

1. Normed Spaces

In $\mathbb{R}^2$ and $\mathbb{R}^3$, we can easily picture the length of a vector. However, when it comes to other, possibly infinite-dimensional vector spaces, we assign a set of axioms to describe the length of a vector in these spaces. These axioms define the "norm" of a vector as we see in the next definition.

DEFINITION 3.1. Let X be a vector space over $\mathbb{F}$. Then a *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for all $x, y \in X$ and $\alpha \in \mathbb{F}$,

(i) $\|x\| \geq 0$;
(ii) $\|x\| = 0$ if and only if $x = 0$;
(iii) $\|\alpha x\| = |\alpha| \|x\|$;
(iv) $\|x + y\| \leq \|x\| + \|y\|$ (the *triangle inequality*).

DEFINITION 3.2. If X is a vector space with a norm on it, then X is a *normed vector space* (also just called a *normed space*).

EXAMPLE 3.3. The function $\|\cdot\| : \mathbb{F}^n \to \mathbb{R}$ defined by

$$\|(x_1, x_2, \ldots, x_n)\| = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2}$$

is the *standard norm* on $\mathbb{F}^k$.

EXAMPLE 3.4. Let M be a compact metric space and recall $C_{\mathbb{F}}(M)$, the set of continuous $\mathbb{F}$-valued functions defined on M. The function $\|\cdot\| : C_{\mathbb{F}}(M) \to \mathbb{R}$ defined by

$$\|f\| = \sup\{|f(x)| : x \in M\}$$

is the *standard norm* on $C_{\mathbb{F}}(M)$.

DEFINITION 3.5. If X is a vector space with norm $\|\cdot\|$, and $d : X \times X \to \mathbb{R}$ is a metric defined by $d(x, y) = \|x - y\|$, then d is called the metric associated with the norm.

THEOREM 3.6. *Let X be a vector space over $\mathbb{F}$ with norm $\|\cdot\|$, $\{x_n\} \in X$ and $\{y_n\} \in X$ be sequences such that $x_n \to x$ and $y_n \to y$ in X, and $\{\alpha_n\} \in \mathbb{F}$ be a sequence such that $\alpha_n \to \alpha$ in $\mathbb{F}$. Then we have the following properties:*

(i) $|\|x\| - \|y\|| \leq \|x - y\|$;

(ii) $\lim_{n\to\infty} \|x_n\| = \|x\|$;
(iii) $\lim_{n\to\infty} (x_n + y_n) = x + y$;
(iv) $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

PROOF.

(i) From the triangle inequality, $\|x\| = \|(x-y)+y\| \le \|x-y\| + \|y\|$. Rearranging, we get $\|x\| - \|y\| \le \|x-y\|$. Similarly for y, $-\|y-x\| \le \|x\| - \|y\|$. However, note that $\|y-x\| = \|(-1)(x-y)\| = \|x-y\|$. Thus we have

$$-\|x-y\| \le \|x\| - \|y\| \le \|x-y\|,$$

and so $|\|x\| - \|y\|| \le \|x-y\|$.

(ii) We know that $x_n \to x$, and from (i), $|\|x\| - \|x_n\|| \le \|x - x_n\|$ for all $n \in \mathbb{N}$. So, letting $n \to \infty$ we get $|\|x\| - \|x_n\|| \le 0$, which means $|\|x\| - \|x_n\|| = 0$ or $\lim_{n\to\infty} \|x_n\| = \|x\|$.

(iii) We know that $x_n \to x$ and $y_n \to y$. By the triangle inequality, $\|(x_n+y_n) - (x+y)\| = \|(x_n - x) + (y_n - y)\| \le \|x_n - x\| + \|y_n - y\|$. Again, letting $n \to \infty$, $\|(x_n + y_n) - (x+y)\| \le 0$, hence $\lim_{n\to\infty} (x_n + y_n) = x + y$.

(iv) Since $\{\alpha_n\}$ is convergent, there exists a $K > 0$ such that $|\alpha_n| \le K$ for all $n \in \mathbb{N}$. Furthermore, we see that $\|\alpha_n x_n - \alpha x\| = \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\| \le |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\| \le K\|x_n - x\| + |\alpha_n - \alpha|\|x\|$. Now since $\alpha_n \to \alpha$ and $x_n \to x$, letting $n \to \infty$ gives $\|\alpha_n x_n - \alpha x\| \le 0$, hence $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

□

DEFINITION 3.7. Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two different norms on X. Then $\|\cdot\|_1$ is *equivalent* to $\|\cdot\|_2$ if there exists an $M, m > 0$ such that for all $x \in X$,

$$m\|\cdot\|_2 \le \|\cdot\|_1 \le M\|\cdot\|_2$$

Note that the notion "equivalent to" defines an equivalence relation on the set of all norms on X.

THEOREM 3.8. *Let X be a finite-dimensional vector space with norm $\|\cdot\|$ and basis $\{e_1, e_2, \ldots, e_n\}$. Define another norm on X by $\|\sum_{j=1}^n \lambda_j e_j\|_1 = \left(\sum_{j=1}^n |\lambda_j|^2\right)^{1/2}$. Then the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.*

PROOF. Let $M = \left(\sum_{j=1}^{n} \|e_j\|\right)^{1/2}$. Since $\{e_1, e_2, \ldots, e_n\}$ forms a basis for X, we have $M > 0$. Additionally,

$$\begin{aligned}\|\sum_{j=1}^{n} \lambda_j e_j\| &\leq \sum_{j=1}^{n} \|\lambda_j e_j\| \\ &= \sum_{j=1}^{n} |\lambda_j| \|e_j\| \\ &\leq \left(\sum_{j=1}^{n} |\lambda_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} \|e_j\|^2\right)^{1/2} \\ &= M\|\sum_{j=1}^{n} \lambda_j e_j\|_1.\end{aligned}$$

Now define a function $f : \mathbb{F}^n \to \mathbb{F}$ by $f(\lambda_1, \ldots, \lambda_n) = \|\sum_{j=1}^{n} \lambda_j e_j\|$, and take $S = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{F}^n : \sum_{j=1}^{n} |\lambda_j|^2 = 1\}$. Now S is compact, so there exists $(\mu_1, \mu_2, \ldots, \mu_n) \in S$ such that $m = f(\mu_1, \mu_2, \ldots, \mu_n) \leq f(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S$. If we have $m = 0$, then $\|\sum_{j=1}^{n} \mu_j e_j\| = 0$, meaning $\sum_{j=1}^{n} \mu_j e_j = 0$. However, this contradicts the fact that $\{e_1, e_2, \ldots, e_n\}$ is a basis for X, thus $m > 0$. Furthermore, by the construction of $\|\cdot\|_1$, if $\|x\|_1 = 1$ then $\|x\| \geq m$. So for $y \in X \setminus \{0\}$ (so that $\left\|\frac{y}{\|y_1\|}\right\| = 1$), we'd have $\left\|\frac{y}{\|y_1\|}\right\| \geq m$, and $\|y\| \geq m\|y\|_1$. For the case $y = 0$ we obviously have $\|y\| \geq m\|y\|_1$. Hence $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. □

LEMMA 3.9. *Let X be a vector space and $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$ be norms on X such that $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$ and $\|\cdot\|_3$ is equivalent to $\|\cdot\|_2$. Then:*

(i) *$\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$;*
(ii) *$\|\cdot\|_3$ is equivalent to $\|\cdot\|_1$.*

PROOF. From the definition of equivalency, there exists an $M, m > 0$ such that $m\|\cdot\|_1 \leq \|\cdot\|_2 \leq M\|\cdot\|_1$, and there exists a $K, k > 0$ such that $k\|\cdot\|_2 \leq \|\cdot\|_3 \leq M\|\cdot\|_2$ for all $x \in X$. So:

(i) Letting $N = \frac{1}{M}$ and $n = \frac{1}{m}$ gives $N, n > 0$ such that $N\|\cdot\|_2 \leq \|\cdot\|_1 \leq n\|\cdot\|_2$ for all $x \in X$. Thus $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$.
(ii) Letting $L = KM$ and $l = km$ gives $L, l > 0$ such that $l\|\cdot\|_1 \leq \|\cdot\|_3 \leq L\|\cdot\|_1$ for all $x \in X$. Hence $\|\cdot\|_3$ is equivalent to $\|\cdot\|_1$.

□

COROLLARY 3.10. *If $\|\cdot\|$ and $\|\cdot\|_2$ are any two norms on a finite-dimensional vector space X, then they are equivalent.*

PROOF. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for X, and let $\|\cdot\|_1$ be the norm defined in Theorem 3.8. Then by Theorem 3.8, both $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent to $\|\cdot\|_1$, and by Lemma 3.9, $\|\cdot\|_2$ is equivalent to $\|\cdot\|$. □

THEOREM 3.11. *Let X be a vector space and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on X such that d and d_1 are metrics defined by $d(x, y) = \|x-y\|$ and $d_1(x, y) = \|x-y\|_1$. Then (X, d) is complete if and only if (X, d_1) is complete.*

A proof of this theorem can be found in [**1**]. Based on this idea, when we have two equivalent norms on the same metric space, we may use whichever is easier to work with to show that the space is complete.

LEMMA 3.12. *Let X be a finite-dimensional vector space over $\mathbb{F}$ with basis $\{e_1, e_2, \ldots, e_n\}$. If $\|\cdot\|_1$ is a norm on X defined by $\|\sum_{j=1}^n \lambda_j e_j\|_1 = \left(\sum_{j=1}^n |\lambda_j|^2\right)^{1/2}$, then X is a complete metric space.*

PROOF. Let $\{x_m\}_{m=1}^\infty$ be a Cauchy sequence in X, and let $\epsilon > 0$. Then each element of the sequence can be written as $x_m = \sum_{j=1}^n \lambda_{j,m} e_j$ for some $\lambda_{j,m} \in \mathbb{F}$. Since $\{x_m\}_{m=1}^\infty$ is Cauchy, there exists an $N \in \mathbb{N}$ such that for $k, m \geq N$,

$$\sum_{j=1}^n |\lambda_{j,k} - \lambda_{j,m}|^2 = \|x_k - x_m\|_1^2 \leq \epsilon^2.$$

So we get $|\lambda_{j,k} - \lambda_{j,m}|^2 \leq \epsilon^2$ for all $k, m \geq N$ and $1 \leq j \leq n$. Thus $\{\lambda_{j,m}\}_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$. Now since $\mathbb{F}$ is a complete metric space, there exists a $\lambda_j \in \mathbb{F}$ such that $\lambda_{j,m} \to \lambda_j$ as $m \to \infty$. By the definition of sequence limits, there exists an $N_j \in \mathbb{N}$ such that for all $m \geq N_j$,

$$|\lambda_{j,m} - \lambda_j|^2 \leq \frac{\epsilon^2}{n}.$$

Now let $N_0 = \max(N_1, N_2, \ldots, N_n)$ and write $x = \sum_{j=1}^n \lambda_j e_j$. Then for $m \geq N_0$,

$$\|x_m - x\|_1^2 = \sum_{j=1}^n |\lambda_{j,m} - \lambda_j|^2 \leq \sum_{j=1}^n \frac{\epsilon^2}{n} = \epsilon^2.$$

Hence by definition, $\lim_{m\to\infty} x_m = x$, and since $\{x_m\}_{m=1}^\infty$ was arbitrary, X is complete.

□

COROLLARY 3.13. *If $\|\cdot\|$ is any norm on a finite dimensional space X, then X is a complete metric space.*

PROOF. Let $\{e_1, e_2, \ldots e_n\}$ be a basis for X and let $\|\cdot\|_1$ be the norm on X defined in Lemma 3.11. Then by Corollary 3.10, the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent, and by Lemma 3.12 the metric space X under the norm $\|\cdot\|_1$ is complete. Thus by Theorem 3.11, X with norm $\|\cdot\|$ is also complete. □

2. Banach Spaces

When we look at infinite-dimensional vector spaces, we find that there may be two norms which are not equivalent. So, we may not assume that theorems from the previous section hold, and will now see what we can say about infinite-dimensional vector spaces.

THEOREM 3.14. *Let X be a normed vector space and S a linear subspace of X. Then $\bar{S}$ is a linear subspace of X.*

PROOF. Take $x, y \in \bar{S}$ and $\alpha \in \mathbb{F}$. Now since x and y are in $\bar{S}$, by Definition 2.10 (iv), there exist sequences $\{x_n\}$ and $\{y_n\}$ in S such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. We know that S is a linear subspace, and so $x_n + y_n \in S$ for all $n \in \mathbb{N}$. Hence

$$x + y = \lim_{n\to\infty} (x_n + y_n) \in \bar{S}.$$

Similarly, $\alpha x_n \in S$ for all $n \in \mathbb{N}$, so

$$\alpha x = \lim_{n\to\infty} \alpha x_n \in \bar{S}.$$

Thus by the subspace test, $\bar{S}$ is a linear subspace of S (and consequently, of X). □

DEFINITION 3.15. A *Banach space* is a normed vector space which is complete under the metric associated with the norm.

THEOREM 3.16. *It is convenient to list a few general spaces which are Banach spaces.*

(i) *Any finite-dimensional normed space is a Banach space.*
(ii) *If M is a compact metric space, then $C_{\mathbb{F}}(M)$ is a Banach space.*
(iii) *If (X, Σ, μ) is a measure space, then $L^p(X)$ is a Banach space (for $1 \leq p \leq \infty$).*
(iv) *ℓ^p is a Banach space (for $1 \leq p \leq \infty$).*
(v) *If X is a Banach space and Y a linear subspace of X, then Y is a Banach space if and only if Y is closed in X.*

PROOF.

(i) This follows from Corollary 3.13.
(ii) This follows from Theorem 2.19.
(iii) This follows from Theorem 2.35.
(iv) This is a special case of part (iii) (by taking counting measure on $\mathbb{N}$).
(v) This proof can be found in [**1**].

□

3. Inner Product Spaces

Recall that in R^3, we can calculate the angle θ between two vectors using the scalar product. Much like we did with the length of a vector and the norm, we can extend this idea to other spaces with higher dimensions by finding a set of axioms that the scalar product satisfies, and using them to form a definition for the general case.

Definition 3.17. Let X be a real vector space. An *inner product* on X is a function $(\cdot,\cdot) : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$,

(i) $(x, x) \geq 0$;
(ii) $(x, x) = 0$ if and only if $x = 0$;
(iii) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$;
(iv) $(x, y) = (y, x)$.

Example 3.18. The function $(\cdot,\cdot) : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ defined by $(x, y) = \sum_{n=1}^{k} x_n y_n$ is the *standard inner product* on $\mathbb{R}^k$.

Note that the above definition and example only apply to real vector spaces. We have a separate definition for complex inner products that is slightly different.

Definition 3.19. Let X be a complex vector space. An *inner product* on X is a function $(\cdot,\cdot) : X \times X \to \mathbb{C}$ such that for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$,

(i) $(x, x) \in R$ and $(x, x) \geq 0$;
(ii) $(x, x) = 0$ if and only if $x = 0$;
(iii) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$;
(iv) $(x, y) = \overline{(y, x)}$.

Example 3.20. The function $(\cdot,\cdot) : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}$ defined by $(x, y) = \sum_{n=1}^{k} x_n \overline{y}_n$ is the *standard inner product* on $\mathbb{C}^k$.

Definition 3.21. If X is a real or complex valued vector space with an inner product $(\cdot,\cdot)$, then X is an *inner product space*.

We notice that since the definitions are so similar, we will continue to generalize theorems and ideas of inner product spaces to $\mathbb{F}$, differentiating between complex and real inner products when necessary. Generally, an inner product can be defined on any finite-dimensional vector space.

Example 3.22. Let X be a k-dimensional vector space with basis $\{e_1, e_2, \ldots, e_k\}$, and $x, y \in X$ such that $x = \sum_{n=1}^{k} \lambda_n e_n$ and $y = \sum_{n=1}^{k} \mu_n e_n$. Then the function $(\cdot,\cdot) : X \times X \to \mathbb{F}$ defined by $(x, y) = \sum_{n=1}^{k} \lambda_n \overline{\mu}_n$ is an inner product on X. Furthermore, if $\{e_1, e_2, \ldots, e_k\}$ is the standard basis on X, then we obtain a standard inner product.

Definition 3.23. The norm $\|\cdot\| : X \to \mathbb{R}$ defined by $\|x\| = (x, x)^{1/2}$ on the inner product space X is known as the *induced norm* of the inner product $(\cdot,\cdot)$.

So every inner product space can be regarded as a normed space using the induced norm. However, it is not true that *every* norm is induced by an inner product.

Theorem 3.24. *Let X be an inner product space and $x, y \in X$. Then the Cauchy-Schwarz inequality says that*

$$|(x, y)| \leq \|x\|\|y\|.$$

Lemma 3.25. *Let X be an inner product space and let $u, v \in X$. If $(x, u) = (x, v)$ for all $x \in X$, then $u = v$.*

4. Hilbert Spaces

When discussing normed spaces, we noted that completeness is an important property (if a normed space is complete, we call it a Banach space). We can also define a complete inner product space as having special characteristics.

DEFINITION 3.26. A *Hilbert space* is an inner product space which is complete with respect to the metric associated with the norm induced by the inner product.

THEOREM 3.27. *Again, we will list some general spaces which are Hilbert spaces.*

(i) *Any finite-dimensional inner product space is a Hilbert space.*
(ii) $L^2(X)$ *with the standard inner product is a Hilbert space.*
(iii) ℓ^2 *with the standard inner product is a Hilbert space.*

PROOF.

(i) This follows from Corollary 3.13.
(ii) This follows from Theorem 2.35.
(iii) This follows from Theorem 2.35.

□

EXAMPLE 3.28. $\mathbb{F}^k$ with the standard inner product is a Hilbert space.

THEOREM 3.29. *If* $\mathcal{H}$ *is a Hilbert space and* $Y \subset \mathcal{H}$ *is a linear subspace, then* Y *is a Hilbert space if and only if* Y *is closed in* $\mathcal{H}$.

PROOF.

($\Rightarrow$) Let $\mathcal{H}$ and Y be as described and assume that Y is a Hilbert space. Then by definition, Y is complete, and by part (i) of Theorem 2.16, Y is closed.

($\Leftarrow$) Assume that $Y \subset \mathcal{H}$ and Y is closed. Since $\mathcal{H}$ is a Hilbert space, it is complete. Hence Y is complete (and a Hilbert space) by part (ii) of Theorem 2.16. □

THEOREM 3.30. *Let* Y *be a closed linear subspace of a Hilbert space* $\mathcal{H}$. *For any* $x \in \mathcal{H}$, *there exists a unique* $y \in Y$ *and* $z \in Y^\perp$ *such that* $x = y + z$, *and* $\|x\|^2 = \|y\|^2 + \|z\|^2$.

A proof of Theorem 3.30 can be found in [**1**]. We will now give a couple definitions and theorems here for use later in the project.

DEFINITION 3.31. Let X be an inner product space. Then the vectors $x, y \in X$ are *orthogonal* if $(x, y) = 0$.

DEFINITION 3.32. Let X be an inner product space with $A \subset X$. The *orthogonal complement* of A is the set

$$A^\perp = \{x \in X : (x, a) = 0 \text{ for all } a \in A\}.$$

So we see that the set $A^\perp$ is the set of vectors in X which are orthogonal to every vector in A (if $A = \emptyset$, then $A^\perp = X$).

EXAMPLE 3.33. If $X = \mathbb{R}^3$ and $A = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$, then $A^\perp = \{(0, 0, x_3) : x_3 \in \mathbb{R}\}$, since by definition, $x \in A^\perp$ if and only if for any $a = (a_1, a_2, 0)$ we have $(x, a) = x_1 a_1 + x_2 a_2 = 0$. If we want this equality to hold for every $a \in A$, we must have $x_1 = x_2 = 0$.

CHAPTER 4

Operators

1. Continuous Linear Transformations

Since we've covered some properties of normed spaces, we can now talk about functions mapping from one normed space into another. The simplest to work with of these maps will be linear (recall the definition of a linear transformation: a function $T : V \to W$ such that for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in V$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$), and the most important will be continuous. We will give examples, as well as look at the space of all continuous linear transformations. First however, we will give alternate characterizations of continuity for linear transformations.

Theorem 4.1. *Let X and Y be normed linear spaces and $T : X \to Y$ be a linear transformation. Then the following are equivalent:*

(i) *T is uniformly continuous;*
(ii) *T is continuous;*
(iii) *T is continuous at 0;*
(iv) *there exists a real number $k > 0$ such that $\|T(x)\| \leq k$ whenever $x \in X$ and $\|x\| \leq 1$;*
(v) *there exists a real number $k > 0$ such that $\|T(x)\| \leq k\|x\|$ for all $x \in X$.*

While the first two implications are quite obvious, the proof of this theorem can be found in [**1**].

Example 4.2. Define $T : C[0,1] \to \mathbb{F}$ by $T(f) = f(0)$. Then if $f \in C[0,1]$, we get

$$\|T(f)\| = |f(0)| \leq \sup\{|f(x)| : x \in [0,1]\} = \|f\|.$$

So using part (v) of Theorem 4.1 with $k = 1$, we see that T is continuous, and also that this was a much easier way of proving continuity than the epsilon-delta methods that we are used to.

Theorem 4.3. *Let X be a finite-dimensional normed space, Y be any normed linear space, and $T : X \to Y$ be a linear transformation. Then T is continuous.*

While we omit the proof here, it can be found in [**1**].

Definition 4.4. Let X and Y be normed linear spaces and $T : X \to Y$ be a linear transformation. T is *bounded* if there exists a real number $k > 0$ such that $\|T(x)\| \leq k\|x\|$ for all $x \in X$.

Notice that this definition is part (v) of Theorem 4.1, hence we can use the words *continuous* and *bounded* interchangeably when talking about linear transformations. However, this use of the word bounded is different than our conventional definition for functions from $\mathbb{R} \to \mathbb{R}$. For example, the linear transformation $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x$ is bounded by Definition 4.4, but obviously not bounded in the usual sense.

Definition 4.5. Let X and Y be normed linear spaces. Then the set of all continuous linear transformations from X to Y is denoted $B(X, Y)$. Elements of this set are called *bounded linear operators* (or sometimes just *operators*).

Lemma 4.6. *Let X and Y be normed linear spaces, and $S, T \in B(X, Y)$ with $\|S(x)\| \leq k_1\|x\|$ and $\|T(x)\| \leq k_2\|x\|$ for all $x \in X$ and some real numbers $k_1, k_2 > 0$. Let $\lambda \in \mathbb{F}$. Then*

(i) $\|(S + T)(x)\| \leq (k_1 + k_2)\|x\|$ *for all* $x \in X$*;*
(ii) $\|(\lambda S)(x)\| \leq |\lambda|k_1\|x\|$ *for all* $x \in X$*;*
(iii) $B(X, Y)$ *is a linear subspace of* $L(X, Y)$*, and so* $B(X, Y)$ *is a vector space.*

Proof.

(i) For $x \in X$,

$$\|(S + T)(x)\| = \|S(x) + T(x)\| \leq \|S(x)\| + \|T(x)\| \leq k_1\|x\| + k_2\|x\| = (k_1 + k_2)\|x\|.$$

(ii) For $x \in X$,

$$\|(\lambda S)(x)\| = |\lambda|\|S(x)\| \leq |\lambda|k_1\|x\|.$$

(iii) First, $B(X, Y)$ is not empty because we can always define We see that $(S + T)$ and (λS) are bounded by Theorem 4.1 (v), and so are elements of $B(X, Y)$. Thus parts (i) and (ii) satisfy the subspace test, so $B(X, Y)$ is a linear subspace of $L(X, Y)$ (and hence is a vector space).

□

For the next definition, recall that if X and Y are normed spaces then the Cartesian product $X \times Y$ is also a normed space.

Definition 4.7. If X and Y are normed spaces and $T : X \to Y$ is a linear transformation, the *graph* of T is the linear subspace of $X \times Y$ defined by

$$\mathcal{G}(T) = \{(x, Tx) : x \in X\}.$$

2. The Space $B(X, Y)$

We just saw that $B(X, Y)$ is a vector space, so we will now look to define a norm. It is important to note that proofs here may have three different norms from different spaces in the same equation. However, we will use the symbol $\|\cdot\|$ as usual for all three norms, as it will be easy to determine which norm we are referring to based on the space that the element is from. We should also note that for a normed linear space X, the set of bounded linear operators from X to X will simply be denoted $B(X)$.

Lemma 4.8. *Let X and Y be normed spaces and define $\|\cdot\| : B(X,Y) \to \mathbb{R}$ by $\|T\| =$ sup $\{\|T(x)\| : \|x\| \leq 1\}$. Then $\|\cdot\|$ is a norm on $B(X,Y)$, so $B(X,Y)$ is a normed space.*

Proof. Before checking the axioms of a norm for $\|\cdot\|$, we must note the following consequence of Theorem 4.1:

$$\sup\{\|T(x)\| : \|x\| \leq 1\} = \inf\{k : \|T(x)\| \leq k\|x\| \ \forall \ x \in X\},$$

$$\text{and so } \|T(y)\| \leq \sup\{\|T(x)\| : \|x\| \leq 1\}\|y\| \ \forall \ y \in X.$$

Now we can proceed checking the axioms of a norm:

(i) Obviously, $\|T\| = \sup \{\|T(x)\| : \|x\| \leq 1\} \geq 0$ for all $T \in B(X,Y)$.
(ii) Note that the zero transformation is defined by $T(x) = 0$ for all $x \in X$. Then

$$\begin{aligned} \|T\| = 0 &\iff \|Tx\| = 0 \ \forall \ x \in X \\ &\iff Tx = 0 \ \forall \ x \in X \\ &\iff T \text{ is the zero transformation.} \end{aligned}$$

(iii) Since $\|T(x)\| \leq \|T\|\|x\|$, we know that $\|(\lambda T)(x)\| \leq |\lambda|\|T\|\|x\|$ for all $x \in X$ by Lemma 4.6 (ii), hence $\|\lambda T\| \leq |\lambda|\|T\|$. If $\lambda = 0$, then clearly $\|\alpha T\| = |\lambda|\|T\|$ and we are done. So assume $\lambda \neq 0$. Then

$$\|T\| = \|\lambda^{-1}\lambda T\| \leq |\lambda^{-1}|\|\lambda T\| \leq |\lambda|^{-1}|\lambda|\|T\| = \|T\|.$$

So we get $\|T\| = |\lambda|^{-1}\|\lambda T\|$, and multiplying both sides by $|\lambda|^{-1}$ gives

$$\|\lambda T\| = |\lambda|\|T\|.$$

(iv) By the construction of $\|T\|$, we see that the triangle inequality holds as follows:

$$\begin{aligned} \|(S+T)(x)\| &\leq \|S(x)\|\|T(x)\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= (\|S\| + \|T\|)\|x\|. \end{aligned}$$

So $\|S+T\| \leq \|S\| + \|T\|$.

Therefore $\|\cdot\|$ is a norm, known as the norm of T. □

Lemma 4.9. *Let X be a normed linear space with W a dense subspace of X, and let Y be a Banach Space with $S \in B(X,Y)$. If $x \in X$ and $\{x_n\}$, $\{y_n\}$ are sequences in W such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$, then $\{S(x_n)\}$ and $\{S(y_n)\}$ both converge, and $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} S(y_n)$.*

Proof. Since $x_n \to x$, $\{x_n\}$ is a Cauchy sequence, and because we have

$$\|S(x_n) - S(x_m)\| = \|S(x_n - x_m)\| \leq \|S\|\|x_n - x_m\|,$$

we see that $\{S(x_n)\}$ is also a Cauchy sequence. Now since Y is a Banach space, it is complete, hence $\{S(x_n)\}$ converges. Furthermore, because of the condition $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$, we must have $\lim_{n\to\infty}(x_n - y_n) = 0$. Now because

$$\|S(x_n) - S(y_n)\| = \|S(x_n - x_m)\| \le \|S\|\|x_n - x_m\|,$$

we get $\lim_{x\to\infty}(S(x_n) - S(y_n)) = 0$, hence $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} S(y_n)$. □

In the next theorem, we explore what conditions are required for $B(X,Y)$ to be a Banach Space.

THEOREM 4.10. *If X is a normed linear space and Y is a Banach space, then $B(X,Y)$ is also a Banach Space.*

PROOF. By the Lemma 4.8, we know that $B(X,Y)$ is a normed space, so we must show that it is complete. Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$. Then $\{T_n\}$ is bounded, so there exists an $M > 0$ such that $\|T_n\| \le M$ for all $n \in \mathbb{N}$. Take $x \in X$. Since we have

$$\|T_n x - T_m x\| = \|(T_n - T_m)(x)\| \le \|T_n - T_m\|\|x\|,$$

and $\{T_n\}$ is Cauchy, it follows that $\{T_n x\}$ is a Cauchy sequence in Y. Now since Y is a Banach space, it is complete, and hence $\{T_n x\}$ is convergent. So we define a transformation $T : X \to Y$ by $T(x) = \lim_{n\to\infty} T_n(x)$. We want to show that T is a bounded linear operator, and that $\{T_n\}$ converges to T. First showing that T is linear, we see

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty}(T_n x + T_n y) = \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty$$

$$\text{and } T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x) = \lim_{n\to\infty} \alpha T_n x = \alpha \lim_{n\to\infty} T_n x = \alpha T(x),$$

so T is a linear transformation. Furthermore, $\|Tx\| = \lim_{n\to\infty}\|T_n x\| \le M\|x\|$, so it follows that T is bounded. Hence $T \in B(X,Y)$. Lastly, we want to show $\lim_{n\to\infty} T_n = T$. So take $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for $m, n \ge N$, $\|T_n - T_m\| < \frac{\epsilon}{2}$. Then for any x with $\|x\| \le 1$ and $m, n \ge N$, $\|T_n x - T_m x\| \le \|T_n - T_m\|\|x\| < \frac{\epsilon}{2}\|x\|$. Now since $T(x) = \lim_{n\to\infty} T_n(x)$, there exists an $N_1 \in \mathbb{N}$ such that for $m \ge N_1$, $\|Tx - T_m x\| < \frac{\epsilon}{2}$. Thus for $n \ge N$ and $m \ge N_1$,

$$\|Tx - T_n x\| \le \|Tx - T_m x\| + \|T_n x - T_m x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}\|x\| \le \epsilon.$$

So we get $\|T - T_n\| \le \epsilon$ when $n \ge N$, thus $\lim_{n\to\infty} T_n = T$. Since $\{T_n\}$ was an arbitrary Cauchy sequence, $B(X,Y)$ is complete. □

DEFINITION 4.11. Let X, Y, and Z be normed linear spaces, $T \in B(X,Y)$ and $S \in B(Y,Z)$. Then the composition of S and T, called the *product* of S and T, will be denoted by ST.

Given the conditions from the definition, if X, Y, and Z are not all the same, the fact that we can define ST does not necessarily mean that we can define TS. However, if $X = Y = Z$, both products will be defined, but generally $TS \neq ST$. Another notation point to make is that if X is a normed space and $T \in B(X)$, the product TT will be denoted T^2, and the product of T with itself n times will be denoted T^n.

3. Isometries, Isomorphisms, and Inverses

In this section we discuss different classifications that may be given to an operator.

Definition 4.12. Let X and Y be normed linear spaces and $T \in L(X,Y)$. If $\|T(x)\| = \|x\|$ for all $x \in X$, then T is called an *isometry*.

One can see it is easy to find the norm of this type of operator. Note that on every normed space, there is at least one isometry, as we see next.

Example 4.13. If X is a normed space and I is the identity linear transformation on X, then I is an isometry since for all $x \in X$, $I(x) = x$. Hence $\|I(x)\| = \|x\|$.

Definition 4.14. If X and Y are normed linear spaces and T is an isometry from X onto Y, then T is called an *isometric isomorphism* and X and Y are *isometrically isomorphic*.

If two spaces are isometrically isomorphic, we can think of them as having essentially the same structure concerning vector space and norm properties. Next we will try and generalize the idea of an inverse to an infinite-dimensional setting.

Definition 4.15. Let X and Y be normed linear spaces. An operator $T \in B(X,Y)$ is *invertible* if there exists an $S \in B(Y,X)$ such that $ST = I_X$ and $TS = I_Y$. We say that S is the *inverse* of T and denote it by T^{-1}.

Definition 4.16. Let X and Y be normed linear spaces. If there exists an invertible operator $T \in B(X,Y)$, then X and Y are *isomorphic*, and we say T is an *isomorphism*.

It is apparent that if $T \in B(X,Y)$ is an isomorphism, then $T^{-1} \in B(X,Y)$ is also an isomorphism. Also, spaces that are isomorphic are similar in structure, as we see next.

Lemma 4.17. *If two normed linear spaces X and Y are isomorphic, then:*

(i) *$dim X < \infty \iff dim Y < \infty$ (in which case, $dim X = dim Y$);*
(ii) *X is separable $\iff$ Y is separable;*
(iii) *X is complete $\iff$ Y is complete.*

Lemma 4.18. *If X and Y are normed linear spaces and $T \in B(X,Y)$ is invertible, then for all $x \in X$,*

$$\|Tx\| \geq \|T^{-1}\|^{-1}\|x\|.$$

Proof. For all $x \in X$ we have $\|x\| = \|T^{-1}(Tx)\| \leq \|T^{-1}\|\|Tx\|$, hence multiplying both sides by the inverse of $\|T^{-1}\|$ gives $\|Tx\| \geq \|T^{-1}\|^{-1}\|x\|$, as desired. □

CHAPTER 5

Duality

1. Dual Spaces

Here we will study another space of functions known as the dual space.

DEFINITION 5.1. Let X be a normed space. Linear transformations from X to $\mathbb{F}$ are called linear *functionals*. The space $B(X,\mathbb{F})$ is called that *dual space* of X and is denoted by X'.

COROLLARY 5.2. *If X is a normed vector space, then X' is a Banach space.*

PROOF. We know that the space $\mathbb{F}$ is complete, hence by Theorem 4.10, $X' = B(X,\mathbb{F})$ is a Banach space. □

DEFINITION 5.3. We define the *Kronecker delta*, denoted δ_{jk}. For any integers j, k,

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}$$

THEOREM 5.4. *If X is a finite dimensional normed linear space with basis $\{v_1, v_2, \ldots, v_n\}$, then X' has a basis $\{f_1, f_2, \ldots, f_n\}$ such that $f_j(v_k) = \delta_{jk}$ for $1 \leq j, k \leq n$. In particular, $dim X' = dim X$.*

PROOF. Let $x \in X$. Since $\{v_1, v_2, \ldots, v_n\}$ is a basis for X, we know there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $x = \sum_{k=1}^{n} \alpha_k v_k$. For $j = 1, \ldots, n$, we define a function $f_j : X \to \mathbb{F}$ by $f_j(x) = \alpha_j$ for $x \in X$. It can be verified that f_j is a linear transformation such that $f_j(v_k) = \delta_{jk}$. Also, f_j is continuous by Theorem 4.3, and so $f_j \in X'$. Now to show that $\{f_1, f_2, \ldots, f_n\}$ is a basis for X', suppose that $\beta_1, \beta_2, \ldots, \beta_n$ are scalars such that $\sum_{j=1}^{n} \beta_j f_j = 0$. Then

$$0 = \sum_{j=1}^{n} \beta_j f_j(v_k) = \sum_{j=1}^{n} \beta_j \delta_{jk} = \beta_k, \quad 1 \leq k \leq n,$$

and so $\{f_1, f_2, \ldots, f_n\}$ is linearly independent.

Now take an arbitrary $f \in X'$ and let $\gamma_j = f(v_j), j = 1, \ldots, n$. Then

$$\sum_{j=1}^{n} \gamma_j f_j(v_k) = \sum_{j=1}^{n} \gamma_k \delta_{jk} = \gamma_k = f(v_k), \quad 1 \leq k \leq n,$$

thus $f = \sum_{j=1}^{n} \gamma_j f_j$, since $\{v_1, v_2, \ldots, v_n\}$ is a basis for X. □

THEOREM 5.5 (Riesz-Frechet Theorem). *Let $\mathcal{H}$ be a Hilbert space and $f \in \mathcal{H}'$. Then there exists a unique $y \in \mathcal{H}$ such that $f(x) = f_y(x) = (x, y)$ for all $x \in \mathcal{H}$. Furthermore, $\|f\| = \|y\|$.*

PROOF. First, we will prove the existence of y. If $f(x) = 0$ for all $x \in \mathcal{H}$, then $y = 0$ works. Otherwise, $\text{Ker}f = \{x \in \mathcal{H} : f(x) = 0\}$ is a proper closed subspace of $\mathcal{H}$, so by Theorem 3.30, $(\text{Ker}f)^{\perp} \neq \{0\}$. Hence there exists a $z \in (\text{Ker}f)^{\perp}$ such that $f(z) = 1$. Now we define $y = \dfrac{z}{\|z\|^2}$ (since $z \neq 0$), and take an arbitrary $x \in \mathcal{H}$. Then since f is linear,

$$f(x - f(x)z) = f(x) - f(x)f(z) = 0,$$

so $x - f(x)z \in \text{Ker}f$. But $z \in (\text{Ker}f)^{\perp}$, so $(x - f(x)z, z) = 0$. Thus $(x, z) - f(x)(z, z) = 0$, so by rearranging we see $(x, z) = f(x)\|z\|^2$, and therefore

$$f(x) = (x, \frac{z}{\|z\|^2}) = (x, y).$$

Now if $\|x\| \leq 1$, then using the Cauchy-Schwarz inequality we get

$$|f(x)| = |(x, y)| \leq \|x\|\|y\| \leq \|y\|,$$

so $f(x) \leq \|y\|$. Furthermore, if $x = \dfrac{y}{\|y\|}$ then $\|x\| = 1$ and

$$\|f\| \geq |f(x)| = \frac{|f(y)|}{\|y\|} = \frac{(y, y)}{\|y\|} = \|y\|.$$

Hence we also have $f(x) \geq \|y\|$, and so $f(x) = \|y\|$.

Next, we will prove uniqueness. If y and w are such that

$$f(x) = (x, y) = (x, w)$$

for all $x \in \mathcal{H}$, then we must have $(x, y - w) = 0$ for all $x \in \mathcal{H}$. So by Lemma 3.25, we have $y - w = 0$, meaning that $y = w$. Hence y is unique. □

Theorem 5.5 gives a representation of all elements of the dual space of a general Hilbert space. So, in a sense, $\mathcal{H}'$ can be identified with $\mathcal{H}$.

2. Sublinear Functionals and Seminorms

If we have a vector space X, many times in functional analysis we will have a linear functional $f_W : W \to \mathbb{F}$ defined on a subspace $W \subset X$, but to use this functional, we need it to be defined on the whole set X. So, we will now see if we can extend the domain from W to X, and get the following definition.

DEFINITION 5.6. Let X be a vector space, W a linear subspace of X, and f_W a linear functional on W. Then a linear functional f_X on X is an *extension* of f_W if $f_X(w) = f_W(w)$ for all $w \in W$.

Definition 5.7. Let X be a real vector space. A *sublinear functional* on X is a function $p : X \to \mathbb{R}$ such that:

(i) $p(x + y) \leq p(x) + p(y) \quad x, y \in X$.
(ii) $p(\alpha x) = \alpha p(x) \quad x \in X, \alpha \geq 0$.

Note that from this definition, it is clear that if p is a sublinear functional on X, then

$$p(0) = 0,$$
$$-p(-x) \leq p(x) \text{ for all } x \in X,$$
$$-p(y - x) \leq p(x) - p(y) \leq p(x - y) \text{ for all } x, y \in X. \tag{2.1}$$

In addition, if we know that p satisfies $p(-x) = p(x)$ for all $x \in X$, then $p(x) \geq 0$ for all $x \in X$, and (1.1) becomes:

$$|p(x) - p(y)| \leq p(x - y) \text{ for all } x, y \in X.$$

Example 5.8. We will list some examples here of sublinear functionals.

(i) If f is a linear functional on X, then it is sublinear.
(ii) If f is a non-zero linear functional on X, then the functional $p(x) = |f(x)|$ is sublinear (but it is not necessarily linear).
(iii) If X is a normed space then $p(x) = \|x\|$ is sublinear.
(iv) If $X = \mathbb{R}^2$ and $p(x_1, x_2) = |x_1| + x_2$, then p is sublinear.

Definition 5.9. Let X be a real or complex vector space. A *seminorm* on X is a real-valued function $p : X \to \mathbb{R}$ such that:

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$;
(ii) $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and $\alpha \in \mathbb{F}$.

Notice the distinctions between the definition of a sublinear functional and seminorm. We have that if X is real, then a seminorm p is a sublinear functional (but the converse is not necessarily true). Also notice that if we add the condition $p(x) = 0$ implies $x = 0$, then the seminorm p meets the criteria of a norm.

Also, the definition implies that if p is a seminorm, then

$$p(0) = 0,$$

$$p(-x) = p(x) \text{ for all } x \in X,$$

$$p(x) \geq 0 \text{ for all } x \in X, \text{ and}$$

$$|p(x) - p(y)| \leq p(x - y) \text{ for all } x, y \in X.$$

3. The Hahn-Banach Theorem

We will state some definitions and lemmas here for use in our proof of the Hahn-Banach theorem. This theorem allows us to construct extensions of functionals on general spaces, and show the existence of elements in a dual space.

DEFINITION 5.10. Suppose that $\mathcal{M}$ is a non-empty set, and $\prec$ is an ordering on $\mathcal{M}$. Then $\prec$ is a *partial order* on $\mathcal{M}$ if

(i) $x \prec x$ for all $x \in \mathcal{M}$;
(ii) $x \prec y$ and $y \prec x \Rightarrow x = y$;
(iii) $x \prec y$ and $y \prec z \Rightarrow x \prec z$;

and we say $\mathcal{M}$ is a *partially ordered set.*

In addition, if $\prec$ is defined for all pairs of elements (so for any $x, y \in \mathcal{M}$, either $x \prec y$ or $y \prec x$ holds), then $\prec$ is a *total order* and $\mathcal{M}$ is a *totally ordered set.*

Furthermore, if $\mathcal{M}$ is a partially ordered set, then $y \in \mathcal{M}$ is a *maximal element* of $\mathcal{M}$ if $y \prec x \Rightarrow y = x$. If $\mathcal{N} \subset \mathcal{M}$, then $y \in \mathcal{M}$ is an *upper bound* for $\mathcal{N}$ if $x \prec y$ for all $x \in \mathcal{N}$.

EXAMPLE 5.11. We use partial and total orders in mathematics all the time.

(i) The usual ordering, $\leq$ on $\mathbb{R}$ is a total order.
(ii) A partial order on $\mathbb{R}^2$ is given by $(x_1, x_2) \prec (y_1, y_2) \iff x_1 \leq y_1$ and $x_2 \leq y_2$.
(iii) If S is an arbitrary set and $\mathcal{M}$ is the set of all subsets of S, then set inclusion ($A \subset B$ for $A, B \in \mathcal{M}$) is a partial order on $\mathcal{M}$.

LEMMA 5.12. *Let X be a real vector space with a proper linear subspace, W. Let p be a sublinear functional on X and f_W be a linear functional on W such that $f_W(w) \leq p(w)$ for all $w \in W$. Suppose that $z_1 \notin W$, and let*

$$W_1 = \mathrm{Sp}\{z_1\} \oplus W = \{\alpha z_1 + w : \alpha \in \mathbb{R}, w \in W\}.$$

Then there exists a $\xi_1 \in \mathbb{R}$ and $f_{W_1} : W_1 \to \mathbb{R}$ that satisfies

$$f_{W_1}(\alpha z_1 + w) = \alpha \xi_1 + f_W(w) \leq p(\alpha z_1 + w), \quad \alpha \in \mathbb{R}, w \in W. \tag{3.1}$$

Since f_W is a linear functional, f_{W_1} is also linear, and for $w \in W$ we have $f_{W_1}(w) = f_W(w)$, so f_{W_1} is an extension of f_W.

PROOF. For any $u, v \in W$, we have

$$f_W(u) + f_W(v) = f_W(u + v) \leq p(u + v) \leq p(u - z_1) + p(v + z_1),$$

which gives

$$f_W(u) - p(u - z_1) \leq -f_W(u) + p(v + z_1).$$

Hence

$$\xi_1 = \inf_{v \in W} \{-f_W(v) + p(v + z_1)\} > -\infty,$$

and

$$-\xi_1 + f_W(u) \leq p(u - z_1), \quad \xi_1 + f_W(v) \leq p(v + z_1), \quad u, v \in W.$$

Now multiplying the first inequality by a $\beta > 0$ gives

$$-\beta\xi_1 + f_W(\beta u) \leq p(\beta(u - z_1)),$$

and if we let $\alpha = -\beta$ and $w = \beta u$, we get the desired inequality (3.1) when $\alpha < 0$. Similarly, the second inequality gives (3.1) when $\alpha > 0$, and for $\alpha = 0$, (3.1) follows immediately from the way p is defined. □

Lemma 5.13 (Zorn's Lemma). *Let $\mathcal{M}$ be a non-empty, partially ordered set such that every totally ordered subset of $\mathcal{M}$ has an upper bound. Then there exists a maximal element in $\mathcal{M}$.*

Theorem 5.14 (The Hahn-Banach Theorem). *Let X be a real vector space, with a sublinear functional p defined on X. Suppose that W is a linear subspace of X, and f_W is a linear functional on W satisfying*

$$f_W(w) \leq p(w), \quad w \in W.$$

Then f_W has an extension f_X on X such that

$$f_X(x) \leq p(x), \quad x \in X.$$

Proof. Let $\mathcal{E}$ denote the set of linear functionals f on X that satisfy the following:

(i) f is defined on a linear subspace D_f such that $W \subset D_f \subset X$;
(ii) $f(w) = f_W(w)$ for $w \in W$;
(iii) $f(x) \leq p(x)$ for $x \in D_f$.

So $\mathcal{E}$ is the set of all extensions f on f_W to general subspaces $D_f \subset X$ which satisfy the hypothesis of the theorem on their domain. We will apply Zorn's lemma to our set $\mathcal{E}$, and then show that the maximal element of $\mathcal{E}$ is our desired functional. So first let us verify that $\mathcal{E}$ satisfies the criteria of Zorn's lemma. We know that $f_W \in \mathcal{E}$, so $\mathcal{E} \neq \emptyset$. Now define a relation $\prec$ on $\mathcal{E}$ as follows: for any $f, g \in \mathcal{E}$,

$$f \prec g \iff D_f \subset D_g \text{ and } f(x) = g(x) \text{ for all } x \in D_f.$$

So $f \prec g$ iff g is an extension of f. One can verify that $\prec$ is a partial order on $\mathcal{E}$ (though it is not a total order, since there can be functionals $f, g \in \mathcal{E}$ which are both extensions of f_W but neither is an extension of the other). Now, take a totally ordered set $\mathcal{G} \subset \mathcal{E}$. By definition, $\mathcal{G}$ is totally ordered iff for any two arbitrary functionals $f, g \in \mathcal{G}$, one is an extension of the other. We will construct an upper bound for $\mathcal{G}$ in $\mathcal{E}$. Define a set

$$Z_{\mathcal{G}} = \bigcup_{f \in \mathcal{G}} D_f.$$

Now using the total ordering of $\mathcal{G}$, it can be verified that Z_g is a linear subspace of X. So we will define a linear functional f_G on $Z_{\mathcal{G}}$ as follows. Given a $z \in Z_{\mathcal{G}}$, there exists a $\xi \in \mathcal{E}$ such that $z \in D_\xi$. Next define $f_{\mathcal{G}}(z) = \xi(z)$ (this definition does not depend on ξ, since if η is a functional in $\mathcal{G}$ with $z \in D_\eta$, we must have $\xi(z) = \eta(z)$ by the total ordering of $\mathcal{G}$). Again, we can verify $f_{\mathcal{G}}$ is linear by the total ordering of $\mathcal{G}$. Also, since $\xi \in \mathcal{E}$, we have $f_{\mathcal{G}}(z) = \xi(z) \leq p(z)$, and if $z \in W$ then $f_{\mathcal{G}}(z) = f_W(z)$. Thus $f_G \in \mathcal{E}$ and $f \prec f_{\mathcal{G}}$ for all $f \in \mathcal{G}$, and we have that $f_{\mathcal{G}}$ is an upper bound for $\mathcal{G}$.

Now since $\mathcal{G}$ was an arbitrary, totally ordered subset of $\mathcal{E}$, we conclude by Zorn's lemma that $\mathcal{E}$ has a maximal element, $f_{\max}$. Now suppose that the domain $D_{f\max} \neq X$. Then by Lemma 5.12, $f_{\max}$ has an extension which also lies in $\mathcal{E}$. However, this would contradict the maximality of $f_{\max}$ in $\mathcal{E}$. So we must have $D_{f\max} = X$, and hence $f_X = f_{\max}$ is our desired extension. □

CHAPTER 6

Concluding Remarks

In this paper, we have given a basic introduction to the branch of mathematics known as linear functional analysis. We discussed the fundamental ideas of normed spaces, inner product spaces, operators, dual spaces, and the Hahn-Banach Theorem. These concepts have applications in many other areas of mathematics, including applied studies such as quantum mechanics and differential equations.

There is plenty of opportunity for further study in this field. While I only proved the Hahn-Banach theorem for real vector spaces, there are other versions with different hypotheses, including a version for normed spaces. This is a tool used in many proofs in all areas of mathematics. Staying in functional analysis, some further topics of study include weak convergence, the adjoint of an operator, compact operators, and spectral theory.

Bibliography

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