Pricing of Options

by
Kaitlyn Hindman

A project submitted to the Department of
Mathematical Sciences in conformity with the requirements
for Math 4301 (Honours Seminar)

Lakehead University
Thunder Bay, Ontario, Canada

Abstract

This paper investigates the Black-Scholes model, which is used to obtain an initial fair price for an option in the stock market. The Black-Scholes partial differential equation will be derived using tools from finance, probability theory, stochastic calculus and partial differential equations.

Acknowledgements

I would like to thank my supervisor, Dr. Fridolin Ting, for all his expertise and patience. As well, I'd like to thank Dr. Razvan Anisca, the course coordinator of Math-4301, for his assistance. Furthermore, I'd like to thank the faculty and students in the Mathematical Sciences department that have supported me on my mathematical journey.

Contents

CHAPTER 1

Introduction

Options are powerful investment tools, which are used in the investment industry daily. An example of a market that uses options is the Chicago Board of Options Exchange, commonly referred to as CBOE. There are two types of options: calls or puts. The Black-Scholes Partial Differential Equation solves for a fair initial price of an option before market forces take over. Fisher Black, Robert Merton, and Myron Scholes established the Black-Scholes Partial Differential Equation in 1969. In 1973, the equation was published and revised by Black and Scholes. In 1997, the Nobel prize in Economics was awarded to Merton and Scholes for the Black-Scholes model. Unfortunately, Fisher Black passed away in 1995.

The equation is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Each term will be defined during the progress of the project.

The book *"Paul Wilmott Introduces Quantitative Finance"* will be used to derive the Black-Scholes Partial Differential Equation in this project. Chapter 2 will introduce and discuss the types of options and their characteristics. In Chapter 3, the Binomial Method for determining an option's initial market price will be explained. Some properties of stocks and stock markets, as well as the application of stochastic calculus to their derivation, will be explored in Chapter 4. The proof of the Black-Scholes Partial Differential Equation is provided in Chapter 5, as well as an application of this model. Finally a short summary of the project will be provided in Chapter 6, with the addition of potential ideas for further study concerning this subject.

CHAPTER 2

Options

Options are financial instruments which allow investors to speculate on future stock prices. In general, profits are made if one's speculation is correct, when excluding all transaction costs. The two types of options are: Calls and Puts. The following definitions and theorems from this chapter were found in "Paul Wilmott Introduces Quantitative Finance". [2]

1. Preliminaries

Definition 2.1. A **call option** gives the holder the right to buy a particular asset for an agreed price on a specified date.

Definition 2.2. A **put option** gives the holder the right to sell a particular asset for an agreed price on a specified date.

Additionally, both these types of options are classified as American or European.

Definition 2.3. An **American** option may be exercised on or before the expiration date.

Definition 2.4. A **European** option is an option that may only be exercised on the expiration date.

For this project all options discussed will be European. Furthermore, options may be characterized as vanilla or binary.

Definition 2.5. A **vanilla** option is the simplest call or put option, with no special features.

Definition 2.6. A **binary** option is either a call or put option, that one purchases with the expectation of a drastic rise or fall of stock value. The payoff at expiration is discontinuous in the underlying asset price, with either "all" the profit or "none" of the profit.

Within this paper all examples of options studied will be vanilla.

The four elements of an option's contract are: Strike Price, Premium, Underlying Asset, and Expiration Date.

Definition 2.7. **Strike Price** is the agreed amount for which the underlying asset can be sold (put) or bought (call). The strike price will be denoted by E.

Definition 2.8. **Premium** is the initial cost of the option contract equal to

$$(Price\ of\ an\ option\ to\ purchase\ one\ share) \times (\ n\ number\ of\ shares).$$

Definition 2.9. An **Underlying Asset** is the financial instrument in which the option value, denoted by S, may be called stock price.

Definition 2.10. The **Expiration Date** is the day at which the option may be exercised, denoted by T.

At the time of expiration, there are three possible outcomes:

Case		Call	Put
1	In The Money	**S >E**	**S <E**
2	At The Money	**S = E**	**S = E**
3	Out of The Money	**S <E**	**S >E**

CASE 1: *"In the Money"* implies that both a call or put option will be exercised and profit exists.

CASE 2: *"Out of the Money"* implies that neither a call or a put option will be exercised and the investor faces the loss of the premium.

CASE 3: *"At the Money"* implies that neither a call or a put option will be exercised and the investor faces the loss of the premium.

From the investigation of each possible case, the payoff functions for a call and put are derived.

Theorem 2.11. *The Payoff Function for a* ***call option*** *is* $max(S - E, 0)$ *per share.*

Theorem 2.12. *The Payoff Function for a* ***put option*** *is* $max(E - S, 0)$ *per share.*

The payoff function does not take into consideration the premium, hence payoff is not equivalent to profit.

Theorem 2.13. *Profit for a* ***call option****, denoted as* π_C *is*

$$[max(S - E, 0) \times (\ n\ number\ of\ shares)] - Premium.$$

Theorem 2.14. *Profit for a* ***put option****, denoted as* π_P *is*

$$[max(E - S, 0) \times (\ n\ number\ of\ shares)] - Premium.$$

Therefore, in general, a call option is profitable if you expect the stock price to rise; a put option is profitable if you expect the stock price to fall.

Here is an example of a call and put option.

Question 2.15. *A European call option is purchased by an investor with a strike price of \$100 to purchase 100 shares of a certain underlying asset. The current underlying asset* *value is \$98. The price of an option to purchase one share is \$1 with an expiration date in 4 months. What is the profit if the stock price at expiry is: 1) \$97 2) \$108.*

Solution: 1)

E= \$100

S= \$97

Premium = (\$1 to purchase one share X 100 shares) = \$100

We can clearly see that, $S < E$ for this call option, and thus the option expires worthless. This implies the payoff of this call option is equal to zero.

Thus, π_C = Payoff - Premium

$= [max(S - E, 0)\, X\, (\ n\ number\ of\ shares)] - Premium$

= 0 - \$100

Therefore π_C = - \$100

Solution: 2)

E= \$100

S= \$108

Premium= \$100

Therefore, since $E < S$, the call option is "in the money" so the call option is exercised.

Payoff of the option = max(S-E,0)

= max(\$108-\$100, \$0)

=max (\$8, \$0)

=\$8 per share

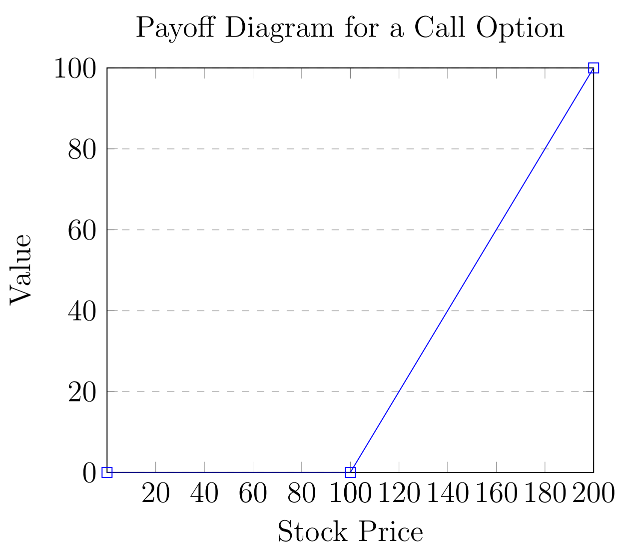
Since the contract was for 100 shares, the payoff of this option was a total of \$800 when the holder of the option purchases 100 shares at the strike price.

Thus, π_C = Payoff - Premium

$=[max(S - E, 0) \times (\ n\ number\ of\ shares)] - Premium$

= \$800 - \$100

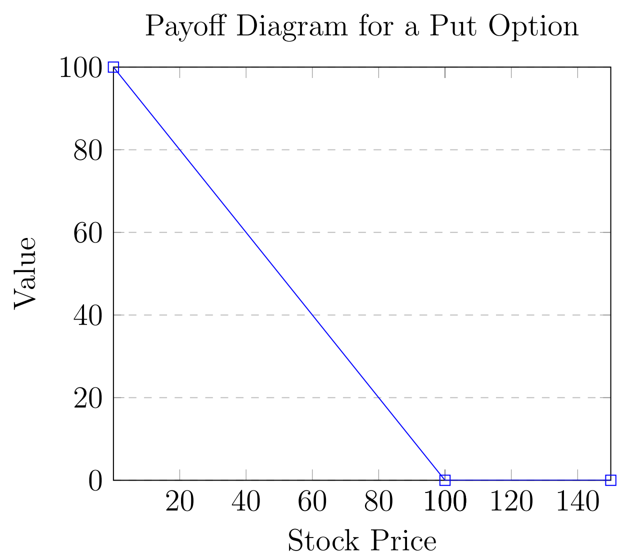
Therefore, π_C = \$700.



QUESTION 2.16. *A European put option is purchased by an investor with a strike price of \$100 to purchase 100 shares of a certain underlying asset. The current underlying asset value is \$102. The price of an option to purchase one share is \$1 with an expiration date in 4 months. What is the profit if the stock price at expiry is: 1) \$108 2) \$95.*

Solution: 1) *E= \$100*
S= \$108
Premium= (\$1 to purchase one share X 100 shares) = \$100
Since E < S for this put option, the option expires worthless. Hence the option has a payoff equal to zero.
Thus, $\pi_P = [max(E - S, 0)\, X\, (\ n\ number\ of\ shares)] - Premium$
π_P *= 0 - \$100*
π_p *= -\$100*

Solution: 2)
E= \$100
S= \$95
Premium= \$100
Since E > S, the put option is "in the money" implies that the put option is exercised.
Payoff of the option = max(E-S,0)
= max(\$100-\$95, \$0)
=max (\$5, \$0)
=\$5 per share
Since the contract was for 100 shares, the payoff of this option was a total of \$500 when the holder of the option sells 100 shares at the strike price.
Thus, π_P*= Payoff - Premium*
$= [max(E - S, 0)\, X\, (\ n\ number\ of\ shares)] - Premium$
= \$500 - \$100
π_P *= \$400*



From the examples, the total loss in the investment is known, where as the profit is unlimited. Note that, options are traded without the necessity to sell or purchase the underlying asset.

2. Application of Options

Many investors choose to deal with options for insurance or to leverage their portfolio.

2.1. Insurance. An investor who owns a substantial amount of a stock would purchase a put option of that stock. The put option minimizes the loss in the investment, but with an initial cost.

QUESTION 2.17. *An investor bought 1000 shares of a stock valued today as $100, as well as a purchase of 1000 put options priced at $1 per share with a strike price of $98. This option will expire in four months. The stock value falls to $90 at the time of expiration. Compare the loss with the consideration of the no option purchase versus an option purchase.*

Solution: *Without the consideration of the option, the investor spends: $100 per share X 1,000 shares = $100,000. However, the stock value falls to $90 now making the value of the stock: $90 X 1,000 shares = $90,000. The total loss in the investment is $10,000.*

With the consideration of the option, the investor spends:

($100 per share X 1,000 shares) + (cost of put option)
=$100,000 + ($1 per share of the option X 1,000 shares)
= $101,000

When the stock falls to $90 but with the option having a strike price of $98 implies that the investor may sell there 1,000 shares of the stock for $98 per share, thus making a gain of $98,000. The investors total loss is $101,000 - $98,000 = $3,000.

Thus, the purchase of a put option saves the investor $7,000 in loss.

2.2. Leverage. Options are associated with leverage by the ability to use a small amount of capital to access a larger amount of capital.

QUESTION 2.18. *The price of a stock is $167 on April 1. The call option that expires August 15 costs $2 per share with a strike price of $171. At the time of expiry the stock value rose to $175. Compare the profit made via buying the stock to that of buying the call option.*

Solution:

Buying the Stock:

With the purchase of $167 initially with a rise to $175 at expiry implies a profit of $8 per stock. In percentage terms, the rate of return is

$$\frac{\$175 - \$167}{\$167} \times 100\% = 4.8\%$$

Buying the Call:
Since $S > E$ at expiry the option is exercised. The investor may purchase the stock valued at \$175 for \$171, which is a profit of \$4 per share. The rate of return is

$$\frac{value\ of\ asset\ at\ expiry -\ strike\ price -\ cost\ of\ call\ option}{cost\ of\ call\ option}$$

$$= \frac{\$175 - \$171 - \$2}{\$2} \times 100\% = 100\%$$

Thus, comparing the two profits it is obvious that the call option created more profit with a small amount of a premium cost versus the purchase of a stock.

A benefit to investing in options is the lower total cost. In the example of leverage, the total cost of purchasing the stock is 83.5 times the cost of an option. Due to the immense difference in total cost of the two methods, investing in options appears to be a more efficient use of capital thereby creating multiple investment opportunities with one's assets.

Options are a financial tool used to speculate on future stock values, insurance for portfolios and to provide a large amount of leverage to one's portfolio. The initial pricing of options before market forces take over is done by the Black-Scholes partial differential equation.

CHAPTER 3

Binomial Method

In 1979, John Cox, Stephen Ross, and Mark Rubinstein established the equations for the binomial method to determine the fair price of an option. Currently, this method is known for its simplicity of calculation, ie. requiring no calculus. The primary sources for this chapter are [1] and [2].

1. Context of the Binomial Method

The Binomial method requires basic arithmetic, with the assistance of tree diagrams. The two important concepts behind the binomial method are:

- no arbitrage principle
- hedging (ie. delta hedging)

DEFINITION 3.1. The **"no arbitrage principle"** refers to the theoretical situation in which one cannot readjust their financial portfolio through different markets to make one's gains increase without an increase in the risk. In other words, there is no such thing as a free lunch.

EXAMPLE 3.2. A Canadian investor purchases a stock on a foreign market. The domestic price for the identical stock is $1 per stock more expensive than the foreign price in the foreign currency. However, due to the exchange rate, it would cost this investor an additional $1 per stock to purchase the foreign stock. Thus, the investor is not able to gain a profit without inducing a risk and the same initial cost of the stock.

DEFINITION 3.3. **Hedging** is a strategy that an investor uses to lower their risk of their investment, with the side effect of a lower profit.

EXAMPLE 3.4. An example of a hedging strategy is when an investor owns n number of shares of a stock, and one decides to purchase a put option on the n number of shares. This will allow the investor to sell the shares of the stock for a known minimum price. This method lowers the risk of one's investment if the stock value does in fact decrease.

DEFINITION 3.5. **Delta hedging** is when the investor's portfolio value is independent of the direction of the value of the stock. In a mathematical formula,

$$\Delta = \frac{range\,of\,option\,payoffs}{range\,of\,stock\,prices}.$$

Simply, Δ is seen as the reactivity of the option with respect to changes in the stock value.

EXAMPLE 3.6. Assume an investor owns n shares of a stock. The investor purchases a call option on that stock. Specifically, assume a stock is priced at \$20 initially. The stock may change by $\pm$\$2. A call option on this stock has a strike price at \$21. If the stock rises to \$22, the payoff of the option is \$1. If the stock price falls, the option expires worthless. Thus the delta of this option is: $\frac{1-0}{22-18} = 0.25$. The investor would delta hedge with the goal of getting $\Delta = 0$. In this case the investor would delta hedge by selling $(number\, of\, options \times |delta| \times n\, shares)$ number of shares.

EXAMPLE 3.7. Similar to the above example, assume an investors portfolio has a delta of -0.25. The investor owns n number of shares of a stock and one put option that generates the delta given. Thus, the investor would delta hedge by buying $(number\, of\, options \times |delta| \times n\, shares) = (1 \times |-0.25| \times n)$ number of shares to readjust delta to 0.

With the combination of both concepts, the following terms are defined:

DEFINITION 3.8. The **time step** over which the asset move takes place, denoted as δt, such that

$$\delta t = \frac{time\, to\, expiry\, in\, terms\, of\, a\, year}{n\, number\, of\, time\, steps} = \frac{T}{n}.$$

DEFINITION 3.9. The rise to a stock value is $\mathbf{u} \times \mathbf{S}$, where as the fall to the stock value is $\mathbf{v} \times \mathbf{S}$, such that $0 < v < 1 < u$.

DEFINITION 3.10. The probability the stock value will **rise** is denoted at p. The probability the stock value will **fall** is equal to $1 - p$.

DEFINITION 3.11. The **drift rate** is the average rate at which the asset rises, denoted as μ.

DEFINITION 3.12. **Volatility** is the measure of an assets randomness in value, denoted as σ.

DEFINITION 3.13. The risk free **interest rate** is the theoretical rate of return an investor would expect from their investment without a risk, which is denoted as r.

Through the association of the discussed terms, the equations regarding the Binomial Method are derived.

PROPOSITION 3.14.

$$u = 1 + \sigma\sqrt{\delta t} \approx e^{\sigma\sqrt{\delta t}}$$

$$v = 1 - \sigma\sqrt{\delta t} = \frac{1}{u}$$

$$p = \frac{1}{2} + \frac{r\sqrt{\delta t}}{\sigma} = \frac{e^{r\delta t} - v}{u - v}$$

$$\boldsymbol{OR}\ p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{\sigma} = \frac{e^{\mu\delta t} - v}{u - v}$$

Remark 3.15. In these equations p, μ and r are interchangeable, and thus it depends on the information provided to determine which equation is applicable.

The binomial method uses backward induction. The notation of this method is as follows.

Definition 3.16. Let f denote the fair value of an option.

Definition 3.17. Let f_u denote the value of an option at the end of a time step in the case that the stock value rose to $u \times S$, with S representing stock price.

Definition 3.18. Let f_v denote the value of an option at the end of a time step in the case that the stock value fell to $v \times S$, with S representing stock price.

Refer to figures 1 and 2 to see the proper placing of each term explained above.

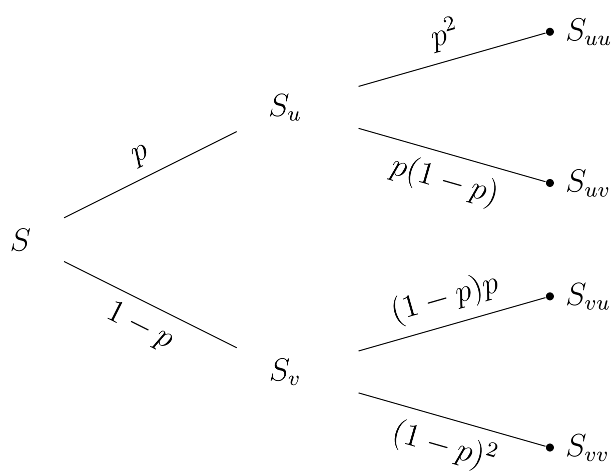


Figure 1. Tree Diagram for the Binomial Method With Two Time Steps

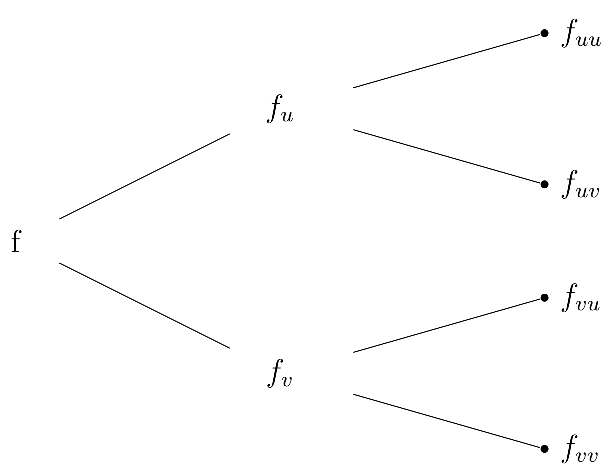


Figure 2. Tree Diagram for the Binomial Method with Two Time Steps

Remark 3.19. The tree diagram grows larger as the number of time steps increases. As well, $S_{uv} = S_{vu}$, $f_{uv} = f_{vu}$, and etc.

Theorem 3.20. *The fair value of an option which is denoted as f satisfies*

$$f = e^{-r\delta t}(pf_u + (1-p)f_v).$$

The formula for solving for f is applied at each individual coinciding branch. The same process is used for any nth time step.

Proposition 3.21. *Applying the binomial method, the following formulas are generated:*
$f = e^{-r\delta t}(pf_u + (1-p)f_v)$,
$f_u = e^{-r\delta t}(pf_{uu} + (1-p)f_{uv})$,
$f_v = e^{-r\delta t}(pf_{uv} + (1-p)f_{vv})$.
This pattern continues for n times.

The steps for using the binomial method to solve for a fair price of an option are:

(1) Use the given values to solve for u, v, and p.
(2) Draw and fill in the appropriate values for the tree diagram. Note that the tree diagram will vary in size depending on the set number of time steps. Refer to figure 1.
(3) Draw a tree diagram of the same size as step 2.
(4) With the knowledge of the option being a call or a put option, at the end of the nth time step, apply the payoff function for each branch of the tree diagram and fill in those values at the terminated nodes.
(5) Using backward induction, apply the formula for solving for f until the beginning node of the tree diagram is reached. This final value of f is the fair price of the option.

2. Applying the Binomial Method

To visualize the binomial method, an example of a European call option will be explored.

Problem 3.22. *What is the fair price of a European call option with a strike price of \$105 and an expiration date in 6 months? Currently, stock price is \$100, volatility is 20% and the risk free interest rate is 10% using three time steps.*

Solution:
E=\$105
S= \$100
σ= *0.2 or 20%*
r= 0.1 or 10%
$\delta t = \frac{T}{n} = \frac{0.5}{3} = \frac{1}{6}$

$u = e^{\sigma\sqrt{\delta t}} = e^{0.2\sqrt{1/6}}$ = *1.08507*
$v = \frac{1}{u} = \frac{1}{1.08507}$ = *0.92159*
$p = \frac{e^{\mu\delta t}-v}{u-v} = \frac{e^{0.1\times 1/6}-0.92159}{1.08507-0.92159}$ = *0.582432*
$S = 100$
$S_u = S \times u = 100 \times 1.08507 = 108.507$
$S_v = S \times v = 100 \times 0.92159 = 92.159$
$S_{uu} = S \times u \times u = 100 \times 1.08507 \times 1.08507 = 117.738$
$S_{uv} = S_{vu} = S \times u \times v = 100 \times 1.08507 \times 0.92159 = 100$
$S_{vv} = S \times v \times v = 100 \times 0.92159 \times 0.92159 = 84.933$
$S_{uuu} = S \times u \times u \times u = 100 \times 1.08507 \times 1.08159 \times 1.08159 = 127.754$
$S_{uuv} = S_{uvu} = S \times u \times u \times v = 100 \times 1.08507 \times 1.08159 \times 0.92159 = 108.507$
$S_{vvu} = S_{vuv} = S \times v \times v \times u = 100 \times 0.92159 \times 0.92159 \times 1.08159 = 92.159$
$S_{vvv} = S \times v \times v \times v = 100 \times 0.92159 \times 0.92159 \times 0.92159 = 78.273$
The results are presented in the tree diagram in figure 3.

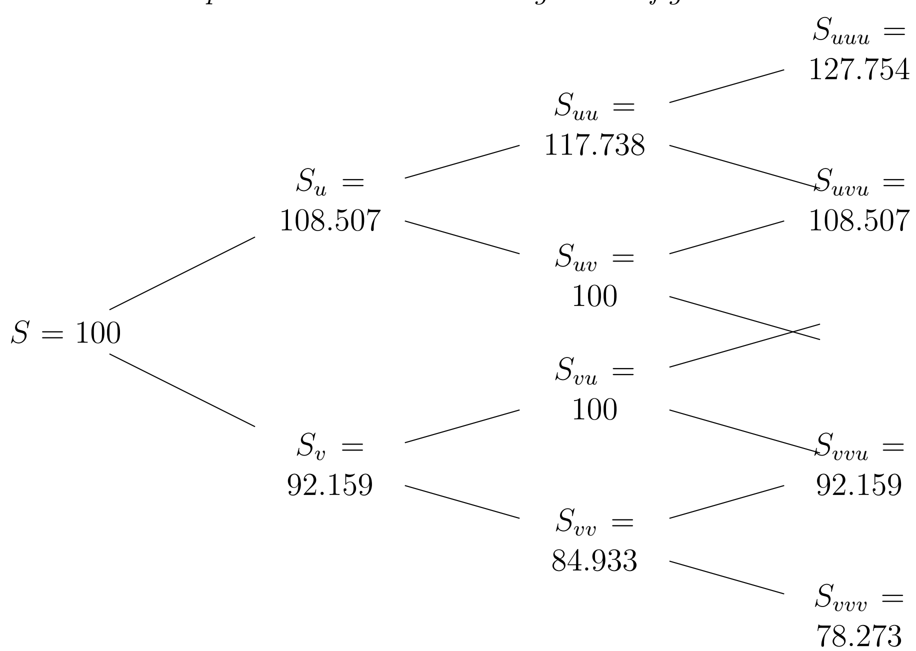


Figure 3. Tree Diagram Showing the Stock Values for Three Time Steps

Apply the payoff functions to the option price tree diagram to solve for the value of the option at the end of the nth term. Therefore in the given problem for the call option, this would be:
f_{uuu}= [max(S_{uuu}- E, 0)] = [max(127.7536 - 105, 0)] = 22.7536
f_{uuv}= [max(S_{uvu} - E, 0)] = [max(108.507 - 105, 0] = 3.507
f_{uvv}= [max(S_{vvu} - E, 0)] = [max(92.159 - 105, 0)] = 0
f_{vvv}= [max(S_{vvv} - E, 0)] = [max(78.273 - 105, 0)] = 0
These results are shown in figure 4.

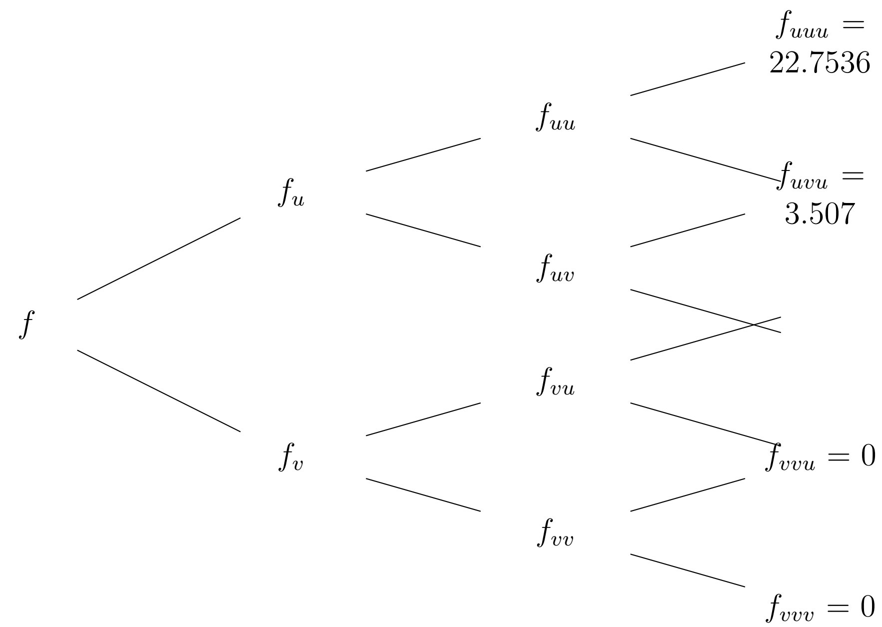


Figure 4. Tree Diagram for the Binomial Method Showing the Value of an Option at the nth Time Step

Thus, from the values found in figure 4, $f_{uu}, f_{uv}, f_{vv}, f_u, f_v,$ *and* f *can be solved for.*

$f_{uu} = e^{-r\delta t}(pf_{uuu} + (1-p)f_{uuv})$
$= e^{-0.1\times 1/6}(0.582432(22.6536) + (1 - 0.582432)(3.507))$
$= 14.4735$
$f_{uv} = e^{-r\delta t}(pf_{uuv} + (1-p)f_{uvv})$
$= e^{-0.1\times 1/6}(0.582432(3.507) + (1 - 0.582432)(0))$
$= 2.008827$
$f_{vv} = e^{-r\delta t}(pf_{uvv} + (1-p)f_{vvv})$
$= 0$
$f_u = e^{-r\delta t}(pf_{uu} + (1-p)f_{uv}) = e^{-0.1\times(1/6)}(0.582432(14.4735) + (1 - 0.582432)2.008827)$
$= 9.115454$
$f_v = e^{-r\delta t}(pf_{uv} + (1-p)f_{vv})$
$= e^{-0.1\times(1/6)}(0.582432(2.008827) + (1 - 0.582432)0)$
$= 1.15066$
Therefore, $f = e^{-r\delta t}(pf_u + (1-p)f_v)$
$= e^{-0.1\times(1/6)}(0.582432(9.115454) + (1 - 0.582432)1.15066)$
$= 5.6939$
The following results are shown in figure 5.

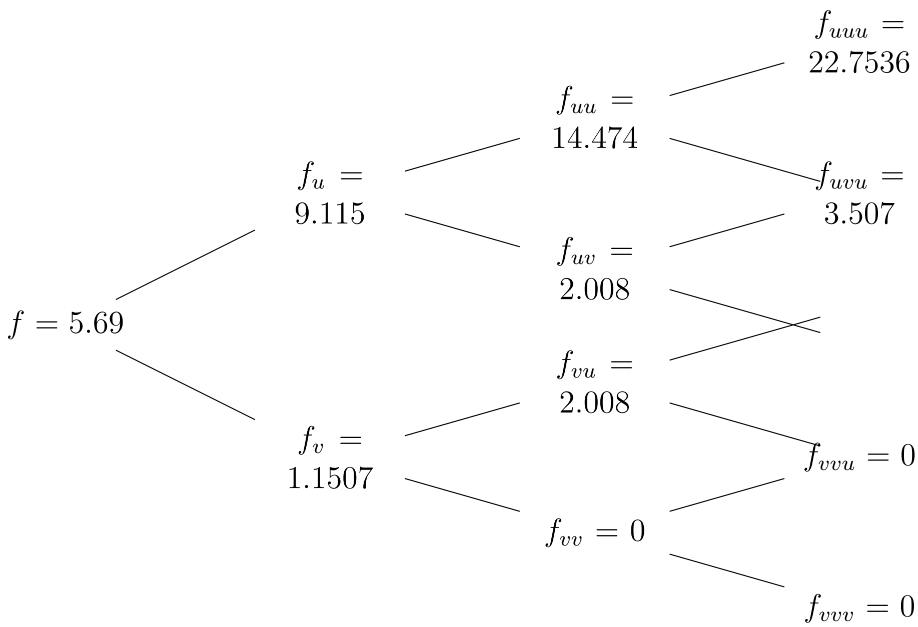


Figure 5. Tree Diagram for the Binomial Method Showing the Value of an Option at Each Time Step

Thus, a fair price for this option is \$5.69.

The binomial method solves for a fair price of an option using simple arithmetic calculations. The Black-Scholes model completes the same objective, using more complicated mathematical concepts. This results in an increase in accuracy of the final solution. The difference in accuracy is due to the fact that the binomial method is based on discrete time, where as Black-Scholes is based on continuous time. Hence there is an accumulation of accuracy via the Black-Scholes model.

CHAPTER 4

The Random Behaviour of Stock Prices and Stochastic Calculus

Asset prices, such as stocks, follow a random pattern that is currently unpredictable. However, certain properties and behaviours concerning specific assets are known, which leads to the derivation of the Black-Scholes Partial Differential equation. The primary sources for this chapter are [2] and [7].

1. Properties and Concepts of Stock Prices

Every investor's goal is to maximize their return in one's investment.

Definition 4.1. Return in a mathematical formula is stated as

$$\frac{change\, in\, value\, of\, the\, asset\, accumulated\, cash\, flows}{original\, value\, of\, the\, asset}.$$

Return is the growth in the value of an asset, in percentage terms. However for this project, dividends are not taken into consideration. Hence,

$$R_i = \frac{S_{i+1} - S_i}{S_i},$$

with R_i denoting return on the ith time value, S_{i+1} is the value of the asset on the $(i+1)$th time value, and S_i is the value of the asset on the ith time value.

While studying the formula for return, similarly as in statistics, the mean and standard deviation of return are derived.

Definition 4.2. The mean of the returns distributions is $\bar{R} = \frac{1}{M}\sum_{i=1}^{M} R_i$, where M is the number is returns in the sample and all other terms are denoted the same as before.

Definition 4.3. Standard Deviation of the sample is $\sqrt{\frac{1}{M-1}\sum_{i=1}^{M}\left(R_i - \bar{R}\right)^2}$.

Using knowledge about statistics, one may assume that the return distribution follows normal distribution and the mean and the standard deviation are both non-zero, constant known values.

Proposition 4.4. *With the assumption that return follows normal distribution, the mathematical formula for return may be written as*

$$R_i = \frac{S_{i+1} - S_i}{S_i} = mean + standard\, deviation \times \phi,$$

with ϕ denoting a standard normal variable.

Theorem 4.5. *The change in the asset value from timestep i to i+1 follows a random walk. Meaning that today's value is known but any value in the future is uncertain. The random walk is displayed in the equation*

$$R_i = S_{i+1} - S_i = \mu S_i \delta t + \sigma \delta t^{1/2} = mean + \phi(standard\, deviation),$$

with $\mu\delta t$ as the mean of the asset value, standard deviation of an asset is denoted as $\sigma\delta t^{1/2}$, and $S_{i+1}, S_i, \delta t, \phi$ follows the notations given previously.

Remark 4.6. This equation is in discrete time. Note that, the proof for Theorem 4.5 is provided in [2].

The familiar terms, such as drift rate and volatility, can be represented in a summation formula when investigating properties of returns.

Proposition 4.7. *μ denotes the drift rate or simply the rate of growth of value for an asset. The formula is stated as,*

$$\mu = \frac{1}{M\delta t}\sum_{i=1}^{M} R_i,$$

where M means the number of revenues in the sample, δt denotes the time step, and R_i is the revenue on the ith time value.

Proposition 4.8. *σ is volatility of the asset and in a mathematical equation is equal to*

$$\sqrt{\frac{1}{(M-1)\delta t}\sum_{i=1}^{M}(R_i - \bar{R})^2},$$

which is most likely not to be constant for any asset. M, R_i, and $\bar{R}$ follow the same notation that was previously given.

Definition 4.9. **Stochastic Differential Equation** is a partial differential equation that follows a stochastic pattern, ie. random and unpredictable.

Definition 4.10. The **Wiener Process** is a primary concept needed to change variables from discrete or normal distribution to continuous time theory.

Question 4.11. *What is the difference between discrete time, normal distribution and continuous time theory?*

Solution:

Discrete: *there is a finite number of distinct time steps between two time values.*

Normal distribution: *this type of probability follows a bell shape pattern, thus using probability one can solve for where a certain value is most likely to be between two real values. Specifically for a function with normal variables, the mean = 0 and the variance is a variable or number value, depending on the equation.*

Continuous: *there is an infinite number of distinct time steps between two time values.*

Continuous time theory has a benefit in most cases as producing a final result with a higher degree of accuracy, thus it is preferred for this project.

Remark 4.12. Let d represent "the change in". For example, dS is the change in the asset value. Now letting $\phi\delta t^{1/2} = dX$, such that dX is a normally distributed term, ie. the mean = 0 and variance = dt. In statistics, this would be written as mean = $E[dX]$ = 0 and variance = $E[dX^2]$ = dt. The above criteria satisfies the Wiener Process.

In finance, one of the most well known stochastic differential equation that follows the Wiener Process format is; $dS = \mu S dt + \sigma S dX$, with dt representing the change in time value. Note that, $\mu S dt$ is the deterministic part and $\sigma S dX$ is the random part of the equation. While studying properties of most financial models, specifically Brownian Motion is investigated.

Definition 4.13. **Brownian Motion** is a stochastic equation denoted as $X(t)$, that has the following main properties:

(1) **Normality:** $X(t)$ is normally distributed for every t, $(X(t) \sim N(\mu = 0, Var(X(t) = t))$.
(2) **Independency:** For each $t_1 < t_2$, $X(t_2) - X(t_1)$ are independent random variables.
(3) **Continuity:** Brownian motion is defined by continuous paths which is no where differentiable.

Example 4.14. An example of a **Stochastic Integral** is

$$W(t) = \int_0^t f(\tau) dX(\tau) = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1})),$$

such that $t_j = \frac{jt}{n}$, and X is Brownian motion.

An important concept that will be expanded on from the lemma is the equation $W(t) = \int_0^t f(\tau) dX(\tau)$. By using basic knowledge of calculus, finding the derivative of $W(t) = \int_0^t f(\tau) dX(\tau)$ is $dW = f(t)\underbrace{dX}_{\text{mean} = 0 \text{ and standard deviation} = \text{d}t^{1/2}}$.

Now with the gained knowledge of Brownian motion, Stochastic calculus and integration methods, Ito's Lemma is derived.

Theorem 4.15. ***Ito's Lemma*** *in integral form is stated as,*

$$F(X(t)) = F(X(0)) + \int_0^t \frac{dF}{dX}(X(\tau)) dX(\tau) + \frac{1}{2}\int_0^t \frac{d^2 F}{dX^2}(X(\tau)) d\tau,$$

or in the differentiated form is

$$dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dt,$$

with F denoting a function, t and τ representing the proper time value (ie. 0 is the initial time value).

Ito's lemma may be applied to different random walk situations. The one that is a major aspect of the Black Scholes model is the lognormal random walk. Specifically, the following proposition gives S(t) in continuous time, using Ito's Lemma.

PROPOSITION 4.16. *The lognormal random walk is represented as*

$$S(t) = S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma(X(t)-X(0))}.$$

PROOF. Assume that an asset follows a simple Brownian motion with drift rate and randomness scale with S, such that $dS = \mu S dt + \sigma S dX$. Note that if S is initially positive it is impossible to become negative. The increments as dS decrease as S goes towards 0. Applying Ito's lemma to $F(S) = logS \Rightarrow$

$$dF = \frac{dF}{dS}dS + \frac{1}{2}\sigma^2 S^2\frac{d^2F}{dS^2}dt = \frac{1}{S}(\mu S dt + \sigma S dX) - \frac{1}{2}\sigma^2 dt = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX.$$

REMARK 4.17. Range of $logS = (-\infty, \infty)$, and range of S is (0,∞) for any finite time interval.

Thus, $dF = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX$
$\Rightarrow$ (from stochastic integration) $S(t) = S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma(X(t)-X(0))}$. □

If a function, denoted as $V(S,t)$, has the property of a lognormal random walk, with the addition of satisfying Ito's lemma, from the proposition above, the following equation is generated;

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2}dt.$$

The equation above looks similar to the Black Scholes partial differential equation with a few minor differences. Thus the lognormal random walk is one of the primary tools used to establishing the Black Scholes model.

CHAPTER 5

The Black Scholes Partial Differential Equation

In this chapter, the Black Scholes equation will be derived. The goal is to have the ability to comprehend and solve for the final value of this formula. The primary sources for this chapter are [2], and [7].

1. Derivation of The Black-Scholes Partial Differential Equation

To begin, set the value of an option equal to $V(S, t; \sigma, \mu; E, T; r)$, with ; separating the different types of values.

Recall that;
S = price of underlying asset
t = time value
σ = volatility
μ =drift rate
E = Strike price
T = expiration date
r = risk free interest rate

Definition 5.1. **Correlation** can be positive or negative and is viewed as the relationship between two or more financial aspects.

For example, when dealing with a call option, there is a positive correlation between the value of an option and the value of the stock price. For a put option, there is a negative correlation between the value of an option and the value of the stock price.

An investor may take a long or short position for an option. Recall that a short and long position on an option is agreeing to sell and buy a specific option contract (ie. specific number of shares), respectively.

If an investor generates a financial portfolio such that one holds one long position and a short position of quantity Δ of the stock, the value of the portfolio denoted Π will be

$$V(S, t) - \Delta S. \quad (1)$$

This specific stock will follow a lognormal random walk, ie. dS = $\mu S dt + \sigma S dX$. Thus the change in the portfolio is partly equal to $d\Pi = dV - \Delta dS$. Applying Ito's lemma, the portfolio will change by

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS.$$

Note that all terms associated with dS are classified as random and dt implies deterministic.

Hedging is the strategy that reduces randomness. Therefore, from equation (1), if randomness is completely eliminated, with $(\frac{\partial V}{\partial S} - \Delta)dS$, then

$$\Delta = \frac{\partial V}{\partial S}. \quad (2)$$

This would be an example of delta hedging, which is also considered a dynamic hedging strategy. If $\Delta = \frac{\partial V}{\partial S}$, then

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt, \quad (3)$$

will be riskless. An example of no arbitrage is

$$d\Pi = r\Pi dt, \quad (4)$$

such that the change in value in a risk free interest account with identical investment is equivalent to the value of $d\Pi$.

The **Black Scholes Equation** is generated when substituting equations (1),(2), and (3) into (4);

$$(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt = r(V - S\frac{\partial V}{\partial S})dt,$$

which can be simplified to,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

This equation is characterized as a linear parabolic partial differential equation.

- linear: if there exists two solutions to this equation, the sum of those two values is also a solution.
- parabolic: this equation is similar to the heat equation, which will be shown in this chapter.

The Black Scholes equation does not take into consideration the drift rate, μ, due to the delta hedging assumption of the model. The following are the assumptions needed for the Black Scholes model:

(1) The underlying asset follows a lognormal random walk.
(2) The risk free interest rate, denoted as r, is a known function of time.
(3) The underlying asset has no dividend.
(4) Delta hedging exists and it is done continuously.
(5) There does not exist any transaction costs on the specific underlying asset.
(6) The ”no arbitrage” principle is held (no arbitrage is considered to be a theoretical condition).

The final condition of the Black Scholes equation is set as the payoff of the option, denoted as $V(S, T)$. The option value V is a function such that S means the value of the underlying asset at the time of expiry, T. Thus the option value in a risk neutral world would be equal to $e^{-r(T-t)}E[Payoff(S)]$.

Note that, the difference between real and risk neutral world is that in the real world the underlying asset follows the actual random walk. In the risk neutral world, there exists a theoretical random walk that an asset may or may not follow. In this case, the drift rate, μ, is equal to the risk free interest rate, r.

Due to the fact that the Black-Scholes equation views time continuously, it implies that the final value of a portfolio remains the same. This means that the value of the portfolio is independent of the movement of the value of the underlying asset.

The Black Scholes equation has the characteristic of a backward equation.

DEFINITION 5.2. **Backward Equation** implies that the first order of the t derivative and the second order of the S derivative, specifically in the Black Scholes equation, will both have the same sign.

THEOREM 5.3.

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty e^{-(log(S/S')+(r-\frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)}Payoff(S')\frac{dS'}{S'}$$

with x'=log S' and V(S,t) denoting the value of an option.

PROOF. First, the value of an option is changed from present to future value terms. Thus,

$$V(S,t) = e^{-r(T-t)}U(S,t).$$

With this change, the differential equation alters to;

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S}.$$

Since Black Scholes equation is a backward equation, set $\tau = T - t$ which implies that

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S}.$$

Assume that a stock follows a lognormal random walk. Let $\xi = \log S$, such that

$$\frac{\partial U}{\partial S} = e^{-\xi}\frac{\partial U}{\partial \xi},$$

then

$$\frac{\partial^2 U}{\partial S^2} = e^{-2\xi}\frac{\partial^2 U}{\partial \xi^2} - e^{2\xi}\frac{\partial U}{\partial \xi}.$$

Hence, the Black Scholes equation is written as

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial \xi}.$$

Note that by the assumption that a stock follows a lognormal random walk, changed the range of the equation from $0 \leq S \leq \infty$ to $-\infty < \xi < \infty$. Also, this is now a constant coefficient partial differential equation.

Through a change of variables, let $x = \xi + (r - \frac{1}{2}\sigma^2)\tau$ and $U = W(x, \tau)$ which refines the Black Scholes equation to

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2}. \quad (5)$$

The goal is to solve for a solution for equation (5) in the form

$$W(x, \tau) = \tau^\alpha f(\frac{(x - x')}{\tau^\beta}) \quad (6)$$

with x' an arbitrary constant.
Substitute equation (6) into (5);

$$\tau^{\alpha-1}(\alpha f - \beta_n \frac{df}{dn}) = \frac{1}{2}\sigma^2 \tau^{\alpha-2\beta} \frac{d^2 f}{dn^2} \quad (7)$$

with $n = \frac{x-x'}{\tau^\beta}$.
A solution for equation (7) is; $\alpha - 1 = \alpha - 2\beta \Leftrightarrow \beta = \frac{1}{2}$.
Verify that $\int_{-\infty}^{\infty} \tau^\alpha f(\frac{x-x'}{\tau^\beta})dx = \int_{-\infty}^{\infty} \tau^{\alpha+\beta} f(n)dn$ which implies that $\alpha = -\beta = -\frac{1}{2}$.
Rewrite the above equation in a differential equation; $-f - n\frac{df}{dn} = \sigma^2 \frac{d^2 f}{dn^2}$ which implies

$$\sigma^2 \frac{d^2 f}{dn^2} + \frac{d(nf)}{dn} = 0. \quad (8)$$

Integrate equation (8);

$$\int (\sigma^2 \frac{d^2 f}{dn^2} + \frac{d(nf)}{dn})dn = \sigma^2 \frac{df}{dn} + nf = a \quad (9)$$

with a denoting a constant.
Set a=0 and integrate equation (9) once again, which will result as $f(n) = be^{-n^2/2\sigma^2}$, with b being a constant that results in $\int_{-\infty}^{\infty} f(n)dn = 1$. Thus, in this case $b = \frac{1}{\sqrt{2\pi\tau}\sigma}$ which implies $f(n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n^2}{2\sigma^2}}$. Thus, this gives us

$$W(x, \tau) = \frac{1}{\sqrt{2\pi\tau}\sigma} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}.$$

Considering that this function has the property of a **Dirac delta function**, ie. $\delta(x' - x)$ as $\tau \to 0$, specifically $\int \delta(x' - x)g(x')dx' = g(x)$ when $x = x' \Rightarrow$

$$\lim_{\tau \to 0} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-(x'-x)^2/2\sigma^2\tau} gx'dx' = g(x).$$

Recall the final condition of the Black Scholes equation is the payoff of the option at expiration, ie. $V(S, T) = Payoff(S)$. Hence, $W(x, 0) = Payoff\ (e^x)$, using the newer

terms discussed.
Let $\tau > 0$, thus the solution to (5) with initial condition $W(x,0) = Payoff(e^x)$ is

$$W(x,\tau) = \int_{-\infty}^{\infty} W_f(x,\tau;x')Payoff(e^{x'})dx'. \quad (10)$$

Substituting the original variables into equation (10) implies that:

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_0^{\infty} e^{-(log(S/S')+(r-\frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)} Payoff(S')\frac{dS'}{S'}$$

with $x' = logS'$, completes this proof. □

Now that a formula for the value of an option is established, the formula can be manipulated to solve for an initial value of a call or put option.

THEOREM 5.4. *The value of a call option is*

$$SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

with $d_1 = \frac{log(S/E)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ *and* $d_2 = \frac{log(S/E)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

PROOF. For a call, the payoff function is; Payoff(S)= max(S-E, 0). Thus, for S', Payoff(S')= $\{S' - E\, for\, S' > E\} \quad OR \quad \{0\, for\, 0 < S' < E\}$.
From theorem 5.3, We get that;

$$V(S',t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_E^{\infty} e^{\frac{-(log\frac{S}{S'}+(r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} \times (S'-E)\frac{dS'}{S'}$$

Setting x' = log S' ⇒

$$= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{logE}^{\infty} e^{-(-x'+log(S)+(r-\frac{1}{2}\sigma^2)(T-t))^2} \times (e^{x'}-E)dx'$$

$$= \tfrac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\textstyle\int_{logE}^{\infty} e^{-(-x'+log(S)+(r-\frac{1}{2}\sigma^2)(T-t))^2} \times (e^{x'})dx'$$
$$- E\tfrac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\textstyle\int_{logE}^{\infty} e^{-(-x'+log(S)+(r-\frac{1}{2}\sigma^2)(T-t))^2}dx' \quad (11)$$

Thus, solving for d_1 and d_2 from equation (11), we conclude that:
$d_1 = \frac{log(S/E)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$
$d_2 = \frac{log(S/E)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.
Finally, the initial call option value is

$$SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

with $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{1}{2}\phi^2}d\phi$. N(x) is simply the area under the curve of normal distribution. □

THEOREM 5.5. *The value of a put option is*

$$-SN(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

with $d_1 = \frac{log(S/E)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ *and* $d_2 = \frac{log(S/E)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

PROOF. The proof is similar for solving the call option value except the payoff of the option is now equal to $max(E-S,0)$, otherwise the steps are almost identical. □

In conclusion, the Black Scholes equation can be manipulated into theorem's 5.3, 5.4, and 5.5, which solve for a fair initial price of an option.

2. The Greeks

In the finance industry, the Black Scholes equation can be represented with the following Greek symbols; $\Delta, \Gamma, \Theta, speed, vega, \rho$, and implied volatility. The following terms will be explained respectively in this order.

DEFINITION 5.6. The **delta** of an option is the sensitivity of the option to the underlying asset. In a mathematical formula, $\Delta = \frac{\partial V}{\partial S}$, with ∂V equal to the rate of change of an option, and ∂S equal to the rate of change of the value of an asset.

REMARK 5.7. In markets that are highly liquid, delta hedging occurs more frequently where as in markets of low liquidity, delta hedging occurs less frequently.

DEFINITION 5.8. A **liquid market** is one that has many buyers and sellers. This means trade occurs quickly and frequently.

THEOREM 5.9. *Delta is equal to:*
1) $\Delta = N(d_1)$ *for a call with a range of (0,1),*
2) $\Delta = N(d_1) - 1$ *for a put with range of (-1, 0),*
such that

$$d_1 = \frac{log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

DEFINITION 5.10. **Gamma**, Γ, of an option is $\frac{\partial^2 V}{\partial S^2}$. Gamma measures the frequency and amount that a position on an option is adjusted to maintain a delta neutral position.

REMARK 5.11. Delta neutral position implies that the overall Δ is equal to 0.

THEOREM 5.12. *Gamma is written as,*

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

for a call and a put, such that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

DEFINITION 5.13. **Theta** is the rate of change of the option price with respect to time, denoted as Θ.

THEOREM 5.14. *Theta is written as:*
1) $-rEe^{-r(T-t)}N(d_2) - \frac{\sigma SN(d_1)}{2\sqrt{T-t}}$ *for a call.*
2) $rEe^{-r(T-t)}N(-d_2) - \frac{\sigma SN(d_1)}{2\sqrt{T-t}}$ *for a put.*

Definition 5.15. The **speed** of an option in a mathematical formula is $\frac{\partial^3 V}{\partial S^3}$. Speed is defined as the rate of change of the gamma with respect to the stock price.

Theorem 5.16. *Speed* $= \frac{-N'(d_1)}{\sigma^2 S^2 (T-t)}[d_1 + \sigma\sqrt{T-t}]$ *for both a call and a put.*

Definition 5.17. **Vega** is the sensitivity of the option price to volatility, written as $\frac{\partial V}{\partial \sigma}$.

Theorem 5.18. *Vega is written as* $S\sqrt{T-t}N'(d_1)$ *for a call and a put.*

Definition 5.19. **Rho** is the sensitivity of the option value to the risk free interest rate that is used in the Black Scholes formula, denoted as $\rho = \frac{\partial V}{\partial r}$.

Theorem 5.20. *Rho is written as:*
1) $E(T-t)e^{-r(T-t)}N(d_2)$ *for a call.*
2) $-E(T-t)e^{-r(T-t)}N(-d_2)$ *for a put.*

Definition 5.21. **Implied volatility** is defined as the volatility of the stock price which when substituted into the Black Scholes equation gives a theoretical price equal to the market price.
To solve for the implied volatility, The Newton-Raphson method is used.

Theorem 5.22. *Implied volatility is denoted as* σ_{n+1} *such that*

$$\sigma_{i+1} = \sigma_i - \frac{V_m - V_t(\sigma_i)}{\frac{dV_t}{d\sigma}(\sigma_i)},$$

with σ_i *denotes an estimated volatility,* V_m *denotes the market value of the option,* V_t *denotes the theoretical value of the option with respect to* σ_i *and* $\frac{dV_t}{d\sigma}$ *denotes vega with respect to* σ_i*, from the Newton Raphson method.*

Note that volatility rarely remains constant. Implied volatility is the market view of volatility, which may in fact not be equal to the exact volatility. Hence, the Black Scholes equation can only solve for the initial fair price of an option. The Black Scholes equation is also written as $\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$, when using the Greek variables.

3. Application of the Black Scholes Partial Differential Equation

Example 5.23. What is the fair price of a European call option with a strike price of \$105 and an expiration date in 6 months. Currently, stock price is \$100, volatility is 20% and the risk free interest rate is 10%.
Solution:
E=\$105
T= 6 months = 0.5
S= \$100
r = 10%
Volatitily= 20%

Therefore, $d_1 = \frac{log(S/E)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = 0.2744$ and $d_2 = \frac{log(S/E)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = 0.1330$. The fair initial value of this call option is $SN(d_1) - Ee^{-r(T-t)}N(d_2) = \$5.59$.

In conclusion, the Black Scholes equation generates a fair initial value for a call and a put, before market forces take over.

CHAPTER 6

Concluding Remarks

The Black Scholes equation is a powerful equation for the financial industry. The equation is able to evaluate the fair initial price of an option before market affects take over. In this paper, the Black Scholes equation is derived using financial, probabilistic, and stochastic integration techniques.

In order to derive the Black Scholes equation, the main concepts and types of options were explained. From this knowledge in combination with delta hedging and the no arbitrage principle, the Binomial method is developed. The Binomial method solves for a fair initial price of an option in discrete time. An exploration of the Wiener process, Brownian motion, Stochastic integration and Ito's Lemma resulted in the necessary background information for deriving the Black Scholes Partial Differential Equation. Using the fact that the Black Scholes model is a Partial Differential Equation, that is characterized as a Backward Heat Equation, we are able to derive a formula that is capable of being numerically evaluated. A brief description of the Greek's shows how the Black Scholes Partial Differential Equation can be represented in terms of the Greek variables. Additionally, an application of the Black Scholes model in the financial industry is given.

In the Future, it would be a great accomplishment to be able to evaluate the value of an option after the market forces take over. For example, being able to numerically analyze the effects of the price of oil based on certain events in the news or changes in the stock market. This quantitative analysis would allow an investor to accurately speculate on the stock market and create an unlimited profit potential.

Bibliography

[1] Hull, John C. *Options, Futures and Other Derivatives.* New Jersey: Pearson Education International, Seventh Edition, 2009.

[2] Wilmott, Paul. *Paul Willmot Introduces Quantitative Finance.* Chichester: John Wiley & Son Ltd, Second Edition, 2007.

[3] Black, F & Scholes, M. 1973 *The Pricing of Options and Corporate Liabilities.* Journal of Political Economy 81: 637-659.

[4] Black F. 1976 *The Pricing of Commodity Contracts.* Journal of Financial Economics 3: 167-179.

[5] ∅ksendal, B. *Stochastic Differential Equations.* Springer-Verlag, 1992.

[6] Wilmott, P. *Paul Wilmott On Quantitative Finance.* second edition. John Wiley & Sons, 2006.

[7] Sauer, Timothy *Numerical Analysis.* Boston: Pearson Education Second Edition, 2006.