The Distribution of Prime Numbers and its Applications

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Abstract

This project will examine the distribution of prime numbers, as well as applications of these results. We begin by approximating how many prime numbers exist that are less than or equal to any given number N. This approximation is known as Tchebychev's Theorem. We then use this result to work through the proof of Mertens' First and Second Theorem. In the proof of Mertens' first Theorem, we show that there exists a bound on the difference between a series of fractions containing primes p, and $\log N$, where p are all the primes less than or equal to N. This difference is proved to be less than 4. We also prove through Mertens' Second Theorem that the difference between the sum of the reciprocals of all the primes less than or equal to a given N and $\ln \ln N$ is also less than a relatively small constant. This constant is independent of N, and can be taken equal to 15.

To my parents,
for their love and support.

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CHAPTER 1

Introduction

Prime numbers have been the focus of mathematicians for centuries. When Euclid's *Elements* were first published circa 300 B.C.E., there were already several important results of prime numbers that had been discovered. In his ninth book, Euclid proved that there are infinitely many prime numbers. In the same book, Euclid also included a proof of the fundamental theorem of arithmetic, which allows us to break a number into a distinct product of its prime factors: an immensely useful tool when working in the area of number theory. For some time after that, the main focuses of prime number theory were developing methods of determining whether a number was prime, and how to find exceedingly larger prime numbers. During this time, the Greek mathematician Eratosthenes developed one of the most effective methods for determining larger primes up to a given limit. However, this method is only efficient for relatively small primes, no higher than 10 million. After this, there was very little development in prime number theory for over a millennium.

Around the beginning of the seventeenth century, Fermat, Mersenne, and Euler made some significant contributions to the field of prime number theory, which included Fermat's Little Theorem as well as the idea of Mersenne Numbers. [**3**] A Mersenne number is one of the form $2^n - 1$, where if n is prime, then the number $2^n - 1$ could potentially be prime. If n was not prime, it was proved that $2^n - 1$ cannot be prime. Throughout the century, Mersenne numbers that were determined to be prime were the largest prime numbers that could be found. For the next few centuries, Cataldi, Euler, and Lucas proved the existence of the Mersenne numbers with $n = 19, 31, 127$, respectively, with centuries between each discovery. To date, 45 Mersenne primes have been discovered, the largest of which with $n = 43112609$. The study of primes naturally led to the study of their distribution, which will be the main focus of this project.

During the late 18th century, both Gauss and Legendre made similar conjectures regarding $\pi(n)$, the number of prime numbers less than or equal to n. In particular, [**2**] in 1798 Legendre estimated that $\pi(x)$ could be estimated to $\frac{x}{\log x - 1.08366}$. It is also impressive that at the age of 15 or 16, Gauss conjectured a similar result to Legendre, which was that $\pi(x) \approx \int_2^x \frac{1}{\log t} dt$. Now we focus our attention to the first person to actually show that $\pi(x) \sim \frac{x}{\log x}$. This result was obtained by Tchebychev in 1852, using completely elementary methods in number theory. The work done by Gauss, Legendre, and especially Tchebychev laid the ground work for solving what is now known as the Prime Number Theorem. The Prime Number Theorem states that as $x \to \infty$, then

$\pi(x) = \frac{x}{\log x}$, which was proved later by Jacques Hadamard and Charles-Jean-Gustave-Nicholas de la Vallée Poussin in 1896 using methods from complex analysis. [**4**] As a result of the Prime Number Theorem, it has been proven that given any $n > 1$, the probability that a randomly selected number x is prime, with $x \leqslant n$, is $\frac{1}{\ln n}$.

1. Preliminaries

For our study of Tchebychev's theorem, we will require some basic knowledge of number theory. This chapter will include many important results that will, in effect, enable us to work through the proof of Tchebychev's theorem later on. It is interesting to note that we can work through a relatively complicated and convoluted proof, with reasonably elementary methods. We begin by formally defining what a prime number is:

Definition 1.1. A number p is **prime** if and only if its only divisors are 1 and itself.

Example 1.2. Given the number $n = 10$, we can clearly break it up into it's prime factors so it will have the form $10 = 2 \cdot 5$. Because 10 is divisible by 1, 2, 5, and 10, we say that 10 *is not* prime.

Example 1.3. Given the number $n = 23$, we can see that 23 cannot be factored further than what it already is, thus only 1 and 23 divide 23. Because *only* 1 and 23 divide 23, we say that 23 *is* prime

Definition 1.4. For $n \in \mathbb{Z}$ we say that n is the **integer part** of a number a if

$$n \leqslant a < n + 1,$$

and we denote it by $n = [a]$.

Example 1.5. We will look at several different cases: $[0.2] = 0$, $[0.9] = 0$, $[3.8] = 3$, $[-3.2] = -4$, $[-5.9] = -6$, $[2] = 2$, and lastly $[-7] = -7$.

Corollary 1.6. *Let $a \in \mathbb{R}$. Then $a - 1 < [a] \leqslant a$ for all a.*

Proof. By the definition, we know that if $n \leqslant a < n+1$, then $n = [a]$. Setting $n = [a]$ in the inequality, we obtain $n = [a] \leqslant a < n + 1$. For the other side of the inequality, we simply subtract 1 from each side of the inequality. This gives us $n - 1 \leqslant a - 1 < n$, and setting $n = [a]$ again, we find that $a - 1 < [a]$. Combining our two inequalities, we obtain $a - 1 < [a] \leqslant a$, our desired result. □

Lemma 1.7. *Given any real number a, then $[a - n] = [a] - n$.*

Proof. Clearly, $[a] - n \leqslant a - n$ because in general $[a] \leqslant a$. We also know that $a - 1 < [a]$ which implies that $a < [a] + 1$. Using this, we find that $a - n < [a] + 1 - n$. Bringing these facts together, we obtain $[a]-n \leqslant a-n < [a]-n+1$. This is the definition of the integer part of $a-n$, thus $[a]-n = [a - n]$, and therefore we obtain the result. □

DEFINITION 1.8. We define an **r-combination** as the number of ways to choose a particular number of elements r from a set of size n, with $n \in \mathbb{Z}$, $n \geqslant 0$, and $0 \leqslant r \leqslant n$. In choosing our r elements, the order of the selected elements does not matter. We say n choose r, and denote it by

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

EXAMPLE 1.9. Suppose we are asked to find the number of possible ways to choose 6 blocks from a group of 10 distinct blocks, regardless of their order. Then the number of ways to do this is equal to

$$C(10,6) = \binom{10}{6} = \frac{10!}{6!(10-6)!} = \frac{10!}{6! \cdot 4!} = 210.$$

Next we will introduce the Binomial Theorem, which helps us begin our proof of Tchebychev's theorem. The proof of the Binomial Theorem is irrelevant to this paper, and is therefore omitted.

THEOREM 1.10. (Binomial Theorem) *Let a and b be variables and let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} a^n b^{n-k} = (a+b)^n.$$

DEFINITION 1.11. We define the **factorial function** as $f : \mathbb{N} \to \mathbb{Z}^+$, denoted by $f(n) = n!$. The value of $f(n) = n!$ is the product of the first n positive integers, so $f(n) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$, and $f(0) = 0! = 1$.

EXAMPLE 1.12. Take 5!. By the definition, $5! = 5 \cdot 4 \cdot\ 3 \cdot\ 2 \cdot 1 = 120$. Similarly, $13! = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6,227,020,800$.

THEOREM 1.13. (Fundamental Theorem of Arithmetic) *Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.*

EXAMPLE 1.14. Take for instance $n = 21$. Then by the Fundamental Theorem of Arithmetic, $21 = 3 \cdot 7$. Similarly, for $n = 420$, we see that $420 = 2^2 \cdot 3 \cdot 5 \cdot 7$.

CHAPTER 2

Tchebychev's Theorem

We begin this section with some necessary definitions and results that will prove to be useful in the proof of Tchebychev's Theorem. Our first definition plays a central role throughout the duration of this paper.

DEFINITION 2.1. Given any number $N > 1$, we define the number of prime numbers less than or equal to N as $\pi(N)$.

LEMMA 2.2. *Let $n, m \in \mathbb{Z}$. Then $\pi(n+m) < \pi(n) + m$.*

PROOF. We know that the number of prime numbers less than or equal to n is $\pi(n)$. Similarly, the number of prime numbers less than or equal to $n+m$ is $\pi(n+m)$. If we take the difference between the number of prime numbers less than $n+m$, and the number of prime numbers less than n, this difference is clearly less than or equal to m. We can see this with the following inequality:

$$\pi(n+m) - \pi(n) \leqslant m.$$

Rearranging, we obtain $\pi(n+m) \leqslant \pi(n) + m$, the required result. □

LEMMA 2.3. *Let b be any real number. Then*

$$2\,[b] = \begin{cases} [2b] & \text{if} \quad b - [b] < \frac{1}{2} \\ [2b] - 1 & \text{if} \quad b - [b] \geqslant \frac{1}{2} \end{cases} \tag{0.1}$$

PROOF. To prove this, we will have to consider two cases, the first of which being if $b - [b] < \frac{1}{2}$. Rearranging, we obtain $b < [b] + \frac{1}{2}$. From here, clearly $2b < 2\,[b] + 1$, and since in general $[b] \leqslant b$, then $2\,[b] \leqslant 2b < 2\,[b] + 1$. This inequality implies that $2\,[b]$ is the integer part of $2b$, and hence $2\,[b] = [2b]$. For the second case, we have that $b - [b] \geqslant \frac{1}{2}$. Rearranging, we obtain $[b] \leqslant b - \frac{1}{2}$. From here, we can see that $2\,[b] \leqslant 2b - 1$, where $2b - 1$ is also equal to $2b - 2 + 1$. Factoring, we obtain $2(b-1) + 1$, and since in general we have that $b - 1 < [b]$, then clearly $2b - 1 = 2(b-1) + 1 < 2\,[b] + 1$. Simplifying the inequality, we find that $2\,[b] \leqslant 2b - 1 < 2\,[b] + 1$. This by definition implies that $2\,[b]$ is the integer part of $2b - 1$, and hence $2\,[b] = [2b - 1]$. Using Lemma (1.7), we have that $2\,[b] = [2b - 1] = [2b] - 1$, which proves the second case. □

LEMMA 2.4. *Given any positive integer N,*

$$N + \frac{N}{2} + \ldots + \frac{N}{2^{k-1}} = \frac{N - \frac{N}{2^k}}{1 - \frac{1}{2}}.$$

Proof. We begin by setting $s_k = N + \frac{N}{2} + \ldots + \frac{N}{2^{k-1}}$. Multiplying s_k by $\frac{1}{2}$, we obtain $\frac{s_k}{2} = \frac{N}{2} + \frac{N}{4} + \ldots + \frac{N}{2^k}$. Now, subtracting $\frac{s_k}{2}$ from s_k, we find that $s_k - \frac{s_k}{2} = N + \frac{N}{2} + \ldots + \frac{N}{2^{k-1}} - \left(\frac{N}{2} + \frac{N}{4} + \ldots + \frac{N}{2^k}\right)$. We can see that all the middle terms of the right hand side cancel, and we are left with $s_k - \frac{s_k}{2} = N - \frac{N}{2^k}$. Factoring out s_k, we have that $s_k\left(1 - \frac{1}{2}\right) = N - \frac{N}{2^k}$, which after further rearrangement gives us $s_k = \frac{N - \frac{N}{2^k}}{1-\frac{1}{2}}$. Now, since $s_k = N + \frac{N}{2} + \ldots + \frac{N}{2^{k-1}}$, we obtain $\left(N + \frac{N}{2} + \ldots + \frac{N}{2^{k-1}}\right) = \frac{N - \frac{N}{2^k}}{1-\frac{1}{2}}$, the result we are looking for. □

With the above information, we now have the necessary tools to prove Tchebychev's Theorem. We will now formally introduce this very important Theorem, and its proof will immediately follow.

Theorem 2.5. (Tchebychev's Theorem) *Let N be any positive integer greater than or equal to 1. Then $\pi(N)$ has the following upper and lower bounds:*

$$0.1\frac{N}{\log N} < \pi(N) < 4\frac{N}{\log N}. \tag{0.2}$$

Proof. Given two numbers a and b, the binomial theorem states that

$$\sum_{k=0}^{n} \binom{n}{k} a^n b^{n-k} = (a+b)^n.$$

Substituting $2n$ for n, and setting both a and b equal to 1, then we have

$$\sum_{k=0}^{2n} \binom{2n}{k} (1)^{2n-k}(1)^k = 1 + \binom{2n}{2} + \binom{2n}{3} + \ldots + \binom{2n}{n} + \ldots + \binom{2n}{2n-1} + 1 = (1+1)^{2n} = 2^{2n}.$$

It is clear that $\binom{2n}{k} > 0$ for any k, thus by rearranging, we can see that

$$\binom{2n}{n} < 2^{2n}. \tag{0.3}$$

We now claim that $\frac{2n-k}{n-k} > 2$ for all $k < n$. Indeed, multiplying both sides by $(n-k)$, which is positive when $k < n$, we obtain

$$\frac{(2n-k)(n-k)}{(n-k)} > 2 \cdot (n-k).$$

Simplifying on the left, and expanding on the right, we obtain $2n - k > 2n - 2k$. After further simplification, we reach our result as it is clear that $2k > k$.

Next, we expand the combination $\binom{2n}{n}$.

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{(n!)^2} \\
&= \frac{2n \cdot (2n-1) \cdot (2n-2) \cdot (2n-3) \ldots (2n-(n-1)) \cdot (2n-n)!}{(n!)^2} \\
&= \frac{2n \cdot (2n-1) \cdot (2n-2) \cdot (2n-3) \ldots (n+1) \cdot (n!)}{(n!)^2} \\
&= \frac{2n \cdot (2n-1) \cdot (2n-2) \cdot (2n-3) \ldots (n+1)}{n!} \\
&= \frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdot \frac{2n-2}{n-2} \cdot \ldots \cdot \frac{n+1}{1} \\
&\geqslant \underbrace{2 \cdot 2 \cdot 2 \ldots 2}_{n \ times} \\
&= 2^n.
\end{aligned}$$

From above, we obtain the following expression

$$2^n \leqslant \binom{2n}{n}. \tag{0.4}$$

Now, from (0.3) and (0.4), we can see that

$$2^n \leqslant \binom{2n}{n} < 2^{2n}. \tag{0.5}$$

We will now look at the prime factorization of $\binom{2n}{n}$. First, we will look at the prime factorization of $k!$ for any positive integer k. Letting p be a prime number less than or equal to k, we can write $k = np^r + \ell$, where p has order r, $n \in \mathbb{N}$, and $\ell < p^r$. Clearly, since $\ell < p^r$, then $k = np^r + \ell < p^r n + p^r = p^r(n+1)$. From here, we obtain the following inequality

$$n = \frac{k-\ell}{p^r} \leqslant \frac{k}{p^r} < n+1,$$

therefore, by definition, n is the integer part of $\frac{k}{p^r}$, thus $n = \left[\frac{k}{p^r}\right]$

Now, expanding $k!$ we obtain the following expression,

$$k! = 1 \cdot 2 \cdot \ldots \cdot p \cdot \ldots \cdot 2p \cdot \ldots \cdot pp \cdot \ldots \cdot 2pp \cdot \ldots \cdot ppp \cdot \ldots \cdot 2ppp \cdot \ldots \cdot k,$$

where p is any prime less than or equal to k. Therefore we obtain $n_1 = \left[\frac{k}{p}\right]$ multiples of p which are less than k, $n_2 = \left[\frac{k}{p^2}\right]$ multiples of p^2 which are less than k, and in general

$n_r = \left[\frac{k}{p^r}\right]$ multiples of p^r which are less than k. This allows us to count all the multiples of p of order 1, as well as all the multiples of p^2 with order 2, and so on, up to all multiples of p^r of order r. For this reason, the power of p in the prime factorization of $k!$ is

$$\alpha = \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \ldots + \left[\frac{k}{p^q}\right],$$

where q is the largest integer such that $p^q < k$

If we now apply this knowledge to the expansion of $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$, we find that it will contain the following,

$$\begin{aligned}
&\frac{p^{\left[\frac{2n}{p}\right]+\left[\frac{2n}{p^2}\right]+\ldots+\left[\frac{2n}{p^q}\right]}}{\left(p^{\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots+\left[\frac{n}{p^q}\right]}\right)^2} \\
&= \frac{p^{\left[\frac{2n}{p}\right]+\left[\frac{2n}{p^2}\right]+\ldots+\left[\frac{2n}{p^q}\right]}}{p^{2\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots+\left[\frac{n}{p^q}\right]\right)}} \\
&= p^{\left(\left[\frac{2n}{p}\right]+\left[\frac{2n}{p^2}\right]+\ldots+\left[\frac{2n}{p^q}\right]\right)-\left(2\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots+\left[\frac{n}{p^q}\right]\right)\right)} \\
&= p^{\left(\left[\frac{2n}{p}\right]-2\left[\frac{n}{p}\right]\right)+\left(\left[\frac{2n}{p^2}\right]-2\left[\frac{n}{p^2}\right]\right)+\ldots+\left(\left[\frac{2n}{p^q}\right]-2\left[\frac{n}{p^q}\right]\right)},
\end{aligned}$$

with p a prime less than or equal to $2n$ and q the largest integer such that $p^q \leqslant 2n$.

Next, using Lemma 2.3, by setting $b = \frac{a}{2}$, we obtain

$$(0.6) \qquad 2\left[\frac{a}{2}\right] = \begin{cases} [a] & \text{if } \frac{a}{2} - \left[\frac{a}{2}\right] < \frac{1}{2} \\ [a]-1 & \text{if } \frac{a}{2} - \left[\frac{a}{2}\right] \geqslant \frac{1}{2} \end{cases}$$

Now, we will examine the expression $[a] - 2\left[\frac{a}{2}\right]$. From (0.8), if $2\left[\frac{a}{2}\right] = [a]$, then $[a] - 2\left[\frac{a}{2}\right] = [a] - [a] = 0$. On the other hand, if $2\left[\frac{a}{2}\right] = [a] - 1$ then $[a] - 2\left[\frac{a}{2}\right] = [a] - ([a] - 1) = 1$. From this, it is clear that

$$(0.7) \qquad [a] - 2\left[\frac{a}{2}\right] \leqslant 1 \quad \text{for all } a.$$

Returning to the following expression,

$$p^{\left(\left[\frac{2n}{p}\right]-2\left[\frac{n}{p}\right]\right)+\left(\left[\frac{2n}{p^2}\right]-2\left[\frac{n}{p^2}\right]\right)+\ldots+\left(\left[\frac{2n}{p^q}\right]-2\left[\frac{n}{p^q}\right]\right)},$$

if we set $a = \frac{2n}{p}$ from (0.7), then $\frac{a}{2} = \frac{n}{p}$. This can therefore be used to show that

$$\left(\left[\frac{2n}{p}\right]-2\left[\frac{n}{p}\right]\right)+\left(\left[\frac{2n}{p^2}\right]-2\left[\frac{n}{p^2}\right]\right)+\ldots+\left(\left[\frac{2n}{p^q}\right]-2\left[\frac{n}{p^q}\right]\right)\leqslant 1+1+\ldots+1=q$$

Now, if we denote by p_i the prime numbers in the prime factorization of $\binom{2n}{n}$, then we can see that each p_i will have a power less than or equal to q_i, where $p_i^{q_i} \leqslant 2n$. This is shown below:

$$\binom{2n}{n} \leqslant p_1^{q_1} \cdot p_2^{q_2} \cdot \ldots \cdot p_r^{q_r} \leqslant \underbrace{2n \cdot 2n \cdot \ldots \cdot 2n}_{r\ times} = (2n)^r.$$

Since there are r many primes less than or equal to $2n$, we can set $r = \pi(2n)$. This means that $\binom{2n}{n} \leqslant 2n^{\pi(2n)}$.

Now, we will take a look back at the following expression:

$$\binom{2n}{n} = \frac{2n \cdot (2n-1) \cdot \ldots \cdot (n+1)}{n \cdot (n-1) \cdot \ldots \cdot 1}.$$

We will take all the primes that divide $\binom{2n}{n}$, and put them into two groups. First of all, we denote the primes that divide $\binom{2n}{n}$ by $p_1, p_2, \ldots, p_s, p_{s+1}, \ldots, p_r$. We will make our first group of primes all the primes less than or equal to n, and they will be the primes $p_1, p_2, \ldots, p_s$. Our second group of primes will consist of all the primes greater than n but less than or equal to $2n$, and they will be the remaining primes; $p_{s+1}, p_{s+2}, \ldots, p_r$. Clearly, the product of all of the primes in our second group divides $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$ since they appear in the numerator only. Thus:

$$\binom{2n}{n} \geqslant p_{s+1} \cdot p_{s+2} \cdot \ldots \cdot p_r > n \cdot n \cdot \ldots \cdot n = n^{r-s}.$$

Now since p_1 up to p_s consist of s many primes, and p_1 to p_r consist of r many primes, then it is clear that the amount of primes from p_{s+1} to p_r is exactly equal to $r-s$. We may also note that the primes p_1 up to p_s are all less than or equal to n as defined above, which by definition means that there are exactly $\pi(n)$ of them. By setting $s = \pi(n)$, we obtain the following expression:

$$(0.8) \qquad n^{r-s} < \binom{2n}{n} \leqslant 2n^{\pi(2n)} \text{ which implies } n^{\pi(2n)-\pi(n)} < \binom{2n}{n} \leqslant 2n^{\pi(2n)}.$$

By comparing (0.5) with the above result, we see that $2^n \leqslant \binom{2n}{n} \leqslant 2n^{\pi(2n)}$ which implies $2^n \leqslant 2n^{\pi(2n)}$. Taking the logarithm of both sides, we find that

$$\begin{aligned}
\log(2^n) &\leqslant \log(2n^{\pi(2n)}) \\
n \cdot \log(2) &\leqslant \pi(2n) \cdot \log(2n) \\
\frac{n \cdot \log(2)}{\log(2n)} &\leqslant \pi(2n) \\
\frac{2n}{\log(2n)} \cdot \frac{\log(2)}{2} &\leqslant \pi(2n) \\
\frac{2n}{\log(2n)} \cdot (0.1505149...) &\leqslant \pi(2n).
\end{aligned}$$

Therefore for any even number $N = 2n$, we have shown the first half of the desired inequality. For N odd, we can use a similar argument to satisfy the inequality. We first claim that $2n \geqslant \frac{2}{3} \cdot (2n+1)$ for $n \geqslant 1$. Clearly, $6n \geqslant 2(2n+1) = 4n+2$. Collecting like terms we find that $2n \geqslant 2$, which is equivalent to $n \geqslant 1$. Now, using this fact, we take any odd $N = 2n+1$. We know from above that $n \cdot \log(2) \leqslant \pi(2n) \cdot \log(2n)$. We can use this to show the following:

$$\begin{aligned}
n \cdot \log(2) \leqslant \pi(2n) \cdot \log(2n) &\leqslant \pi(2n+1) \cdot \log(2n+1) \\
\frac{2}{3} \cdot (2n+1) \cdot \frac{\log(2)}{2} \leqslant 2n \cdot \frac{\log(2)}{2} &\leqslant \pi(2n+1) \cdot \log(2n+1) \\
\frac{(2n+1)}{\log(2n+1)} \cdot \frac{2}{3} \cdot \frac{\log(2)}{2} &\leqslant \pi(2n+1) \\
\frac{(2n+1)}{\log(2n+1)} \cdot (0.1003433...) &\leqslant \pi(2n+1).
\end{aligned}$$

Since for any even N we have $\frac{N}{\log N} \cdot (0.1505149...) \leqslant \pi(N)$, and for any odd N we have $\frac{N}{\log N} \cdot (0.1003433...) \leqslant \pi(N)$, then we can conclude that for all $N > 1$,

$$\frac{N}{\log N} \cdot (0.1003433...) \leqslant \pi(N). \tag{0.9}$$

We now begin the proof of the second half of the desired inequality. From (0.5) and (0.8), we find that $n^{\pi(2n)-\pi(n)} < 2^{2n}$. Taking the logarithm of both sides, we find that

$$\begin{aligned} \log(n^{\pi(2n)-\pi(n)}) &< \log(2^{2n}) \\ (\pi(2n)-\pi(n))\cdot\log n &< 2n\cdot\log 2 \\ \pi(2n)-\pi(n) &< 2\cdot\log 2\cdot\frac{n}{\log n} \\ &= (0.60206)\cdot\frac{n}{\log n}. \end{aligned}$$

We now suppose that $x > 1$, for $x \in \mathbb{R}$. Setting $n = \left[\frac{x}{2}\right]$ and letting $x = a$ from our previous result (0.8), we find that either $[x] = 2n$ or $[x] = 2n+1$. For the first case, if $2\left[\frac{x}{2}\right] = [x]$, then since $n = \left[\frac{x}{2}\right]$, we have that $[x] = 2n$. For the second case, if $2\left[\frac{x}{2}\right] = [x]-1$, then with $n = \left[\frac{x}{2}\right]$, we find that $2n = [x]-1$ which implies that $[x] = 2n+1$.

We now observe two results. The first result is that $\pi(x) = \pi([x]) = \pi(2n+1) \leqslant \pi(2n)+1$, using Lemma (2.2). This is true because $[x]$ is either $2n+1$ or $2n$. The second result is that $\pi(\frac{x}{2}) = \pi(\left[\frac{x}{2}\right]) = \pi(n)$, as $n = \left[\frac{x}{2}\right]$. From these two results, we obtain the following expression:

$$\begin{aligned} \pi(x)-\pi(\frac{x}{2}) &\leqslant \pi(2n)+1-\pi(n) = \pi(2n)-\pi(n)+1 \\ &< 2\cdot\log 2\cdot\frac{n}{\log n}+1 \\ &< 2\cdot\log 2\cdot\frac{n}{\log n}+\frac{n}{\log n} \\ &= (2\cdot\log(2)+1)\cdot\frac{n}{\log n} \\ &= (1.60206)\cdot\frac{n}{\log n}. \end{aligned}$$

We now claim that $\frac{n}{\log n} < \frac{x}{\log x}$ for $n \geqslant 3$ and $n < x$. We begin by looking at the n^{th} root of the first few integers:

$$\sqrt[2]{2} = 1.41 \quad \sqrt[3]{3} = 1.44 \quad \sqrt[4]{4} = 1.41 \quad \sqrt[5]{5} = 1.37 \quad \ldots$$

We can see that for $n \geqslant 3$, $\sqrt[n+1]{n+1} < \sqrt[n]{n}$. Taking the logarithm of both sides, we find that $\frac{1}{n+1}\cdot\log(n+1) < \frac{1}{n}\cdot\log(n)$. This implies that for $x > n$, $\frac{\log x}{x} < \frac{\log n}{n}$. Rearranging, we obtain $\frac{n}{\log n} < \frac{x}{\log x}$. Now, since $n = \left[\frac{x}{2}\right]$, then for $\left[\frac{x}{2}\right] \geqslant 3$,

$$\pi(x)-\pi(\frac{x}{2}) < (2\cdot\log(2)+1)\cdot\frac{x}{\log x}.$$

We can also show that the above inequality holds for $\left[\frac{x}{2}\right] < 3$. First, if $\left[\frac{x}{2}\right] < 3$, this is equivalent to saying $x < 6$. But we know for any $x < 10$, then $\log x < 1$, so clearly

$\frac{1}{\log x} > 1$. Using what we have just shown, we now look at the same expression as before $\pi(x) - \pi(\frac{x}{2})$. Clearly, for all $x > 1$,

$$\pi(x) - \pi\left(\frac{x}{2}\right) < \pi(x) < x < \frac{x}{\log x} < (2 \cdot \log(2) + 1) \cdot \frac{x}{\log x}.$$

We now look at the following inequality:

$$\begin{aligned}
&\pi(x)\log(x) - \pi\left(\frac{x}{2}\right)\log\left(\frac{x}{2}\right) \\
&= \pi(x)\log(x) - \pi\left(\frac{x}{2}\right)\log(x) + \pi\left(\frac{x}{2}\right)\log(x) - \pi\left(\frac{x}{2}\right)\log\left(\frac{x}{2}\right) \\
&= \left[\pi(x) - \pi\left(\frac{x}{2}\right)\right] \cdot \log(x) + \pi\left(\frac{x}{2}\right) \cdot \left[\log(x) - \log\left(\frac{x}{2}\right)\right] \\
&< (2 \cdot \log(2) + 1) \cdot \frac{x}{\log(x)} \cdot \log(x) + \pi\left(\frac{x}{2}\right) \cdot [\log(x) - (\log(x) - \log(2))] \\
&< (2 \cdot \log(2) + 1) \cdot x + \left(\frac{x}{2}\right) \cdot [\log(2)] \\
&= (2 \cdot \log(2) + 1 + \frac{\log(2)}{2}) \cdot x \\
&= (1.75257\ldots) \cdot x.
\end{aligned}$$

Now, taking N to be any arbitrary positive integer, from above we obtain the following set of inequalities:

$$\begin{aligned}
\pi(N)\log(N) - \pi\left(\frac{N}{2}\right)\log\left(\frac{N}{2}\right) &< (1.75257\ldots) \cdot N, \\
\pi\left(\frac{N}{2}\right)\log\left(\frac{N}{2}\right) - \pi\left(\frac{N}{4}\right)\log\left(\frac{N}{4}\right) &< (1.75257\ldots) \cdot \frac{N}{2}, \\
\pi\left(\frac{N}{4}\right)\log\left(\frac{N}{4}\right) - \pi\left(\frac{N}{8}\right)\log\left(\frac{N}{8}\right) &< (1.75257\ldots) \cdot \frac{N}{4}, \\
&\vdots \\
\pi\left(\frac{N}{2^{k-1}}\right)\log\left(\frac{N}{2^{k-1}}\right) - \pi\left(\frac{N}{2^k}\right)\log\left(\frac{N}{2^k}\right) &< (1.75257\ldots) \cdot \frac{N}{2^{k-1}}.
\end{aligned}$$

We now choose k so that $2^k > N$. Adding all the inequalities from the previous step, we obtain the following:

$$\begin{aligned}
&\pi(N)\log(N) - \pi\left(\frac{N}{2}\right)\log\left(\frac{N}{2}\right) + \pi\left(\frac{N}{2}\right)\log\left(\frac{N}{2}\right)\\
&- \pi\left(\frac{N}{4}\right)\log\left(\frac{N}{4}\right) + \ldots + \pi\left(\frac{N}{2^{k-2}}\right)\log\left(\frac{N}{2^{k-2}}\right)\\
&- \pi\left(\frac{N}{2^{k-1}}\right)\log\left(\frac{N}{2^{k-1}}\right) + \pi\left(\frac{N}{2^{k-1}}\right)\log\left(\frac{N}{2^{k-1}}\right) - \pi\left(\frac{N}{2^{k}}\right)\log\left(\frac{N}{2^{k}}\right)\\
&< (1.75257\ldots)\cdot N + (1.75257\ldots)\cdot\frac{N}{2} + \ldots + (1.75257\ldots)\cdot\frac{N}{2^{k-1}}\\
&= (1.75257\ldots)\left(N + \frac{N}{2} + \ldots + \frac{N}{2^{k-1}}\right).
\end{aligned}$$

The above expression can simplify to

$$\pi(N)\log(N) - \pi\left(\frac{N}{2^k}\right)\log\left(\frac{N}{2^k}\right) < (1.75257\ldots)\left(N + \frac{N}{2} + \ldots + \frac{N}{2^{k-1}}\right).$$

Now, using Lemma 2.4, we find that

$$\begin{aligned}
\pi(N)\log(N) - \pi\left(\frac{N}{2^k}\right)\log\left(\frac{N}{2^k}\right) &< (1.75257\ldots)\left(\frac{N - \frac{N}{2^k}}{1-\frac{1}{2}}\right)\\
&= (3.50514)\left(N - \frac{N}{2^k}\right)\\
&< (3.50514)\cdot N < 4N,
\end{aligned}$$

using the fact that $0 < \frac{N}{2^k} < 1$. Setting $\pi(\frac{N}{2^k}) = 0$ due to the fact that $\frac{N}{2^k} < 1$, we can therefore conclude that

$$\begin{aligned}
\pi(N)\log(N) - \pi\left(\frac{N}{2^k}\right)\log\left(\frac{N}{2^k}\right) &< 4N\\
\pi(N)\log(N) - 0\cdot\log\left(\frac{N}{2^k}\right) &< 4N\\
\pi(N)\log(N) &< 4N\\
\pi(N) &< 4\cdot\frac{N}{\log N}.
\end{aligned}$$

Comparing the above inequality with (0.9), we find that

$$(0.1003433...)\cdot\frac{N}{\log N} \leqslant \pi(N) < 4\cdot\frac{N}{\log N},$$

thus completing the proof.

□

We can also complete this proof using the natural logarithm, as opposed to the logarithm of base 10. The proof remains the exact same, except for when computing values in certain parts of the proof. Instead of computing values using log base 10, we can use log base e, which is equivalent to ln. Computing these values, we find that (0.9) can be changed to

$$\frac{N}{\ln N} \cdot (0.2310406) \leqslant \pi(N),$$

and the other side of the inequality can be changed to

$$\pi(N) < 6 \cdot \frac{N}{\ln N}$$

This implies that in terms of the natural logarithm, $\pi(N)$ can be approximated in the following way:

$$\frac{N}{\ln N} \cdot (0.23) \leqslant \pi(N) < 6 \cdot \frac{N}{\ln N}.$$

CHAPTER 3

Mertens' First Theorem

In this section, we use what we have proved in the previous chapter to examine an application of the distribution of prime numbers. We begin by stating some useful tools which will be helpful in the proof Mertens' First Theorem.

LEMMA 3.1. *Let N be an integer and p_i a prime number. Then*

$$\frac{N}{p_i} + \frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i}} = \frac{\frac{N}{p_i} - \frac{N}{p_i^{q_i+1}}}{1 - \frac{1}{p_i}}.$$

PROOF. We begin by setting $s_k = \frac{N}{p_i} + \frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i}}$. Multiplying s_k by $\frac{1}{p_i}$ we obtain $\frac{s_k}{p_i} = \frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i+1}}$. Subtracting s_k from $\frac{s_k}{p_i}$ we obtain

$$\begin{aligned} s_k - \frac{s_k}{p_i} &= \frac{N}{p_i} + \frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i}} - \left(\frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i+1}}\right) \\ &= \frac{N}{p_i} + \frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i}} - \frac{N}{p_i^2} - \frac{N}{p_i^3} - \ldots - \frac{N}{p_i^{q_i}} - \frac{N}{p_i^{q_i+1}} \\ &= \frac{N}{p_i} - \frac{N}{p_i^{q_i+1}}. \end{aligned}$$

Factoring out s_k from the left side, we obtain $s_k(1 - \frac{1}{p_i}) = \frac{N}{p_i} - \frac{N}{p_i^{q_i+1}}$. Dividing both sides by $(1 - \frac{1}{p_i})$, and substituting $\frac{N}{p_i} + \frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i}}$ for s_k, we obtain $\frac{N}{p_i} + \frac{N}{p_i^2} + \frac{N}{p_i^3} + \ldots + \frac{N}{p_i^{q_i}} = \frac{\frac{N}{p_i} - \frac{N}{p_i^{q_i+1}}}{1 - \frac{1}{p_i}}$, the result we are looking for. □

LEMMA 3.2. *Take $N \in \mathbb{N}$, and let $s > 1$ for $s \in \mathbb{R}$. Then*

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots + \frac{1}{N^s}$$

tends to a limit lying between $\frac{1}{s-1}$ and $\frac{s}{s-1}$.

LEMMA 3.3. *Let n be any positive integer. Then $n!$ satisfies the following inequality:*

$$\sqrt{\frac{4}{5}} \cdot e \cdot \sqrt{n} \cdot \left(\frac{n}{e}\right)^n < n! < e \cdot \sqrt{n} \cdot \left(\frac{n}{e}\right)^n.$$

LEMMA 3.4. *Let $a \in \mathbb{R}$. Then*

$$2 \log a < \sqrt{a},$$

for $a > 0$.

PROOF. To show this, we rearrange our expression to look like $\sqrt{a} - 2\log a > 0$. Treating this expression like a function, it is sufficient to show that this function is always positive for $a > 0$. Taking the derivative of the function $f(a) = \sqrt{a} - 2\log a$, we obtain

$$f'(a) = \frac{1}{2\sqrt{a}} - \frac{2\log e}{a} = \frac{a - 4\sqrt{a}(0.43429)}{2a\sqrt{a}} = \frac{\sqrt{a} - 1.73716}{2a}.$$

Setting our derivative equal to 0, we find that

$$0 = \frac{\sqrt{a} - 1.73716}{2a} \Rightarrow \sqrt{a} = 1.73716 \Rightarrow a = 3.01772,$$

therefore we must have a maximum or minimum at a=3.01772, and an asymptote at $a = 0$. Substituting values that are in neighborhood of a=3.01772 into our derivative, we find that $f'(3)$ is negative, and $f'(3.1)$ is positive, meaning that our function decreases to $a = 3.01772$ and then increases indefinitely. This is illustrated by substituting values into our original function:

$$f(3) = 0.777808 \qquad f(3.01772) = 0.777801 \qquad f(3.1) = 0.77795$$

Because $f(3.01772)$ is our local minimum, and it is positive, we can therefore conclude that for $a > 0$, $f(a) = \sqrt{a} - 2\log a$ is positive. This implies that $\sqrt{a} - 2\log a > 0$ and therefore $2\log a < \sqrt{a}$. □

THEOREM 3.5. (Mertens' First Theorem) *Let $2, 3, 5, 7, 11, 13, \ldots, p$ be the primes not exceeding a given integer N. Then there exists a constant R, that can be taken equal to 4, such that*

$$\left| \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 5}{5} + \ldots + \frac{\log p}{p} - \log N \right| < R.$$

PROOF. We begin by decomposing a given integer $N!$ into its prime factors. As we have seen in the proof of Tchebychev's Theorem, $N!$ has the following form:

$$N! = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}, \tag{0.10}$$

with $p_1, p_2, \ldots, p_r$ the primes less than or equal to N, and

$$\alpha_i = \left[\frac{N}{p_i}\right] + \left[\frac{N}{p_i^2}\right] + \left[\frac{N}{p_i^3}\right] + \ldots + \left[\frac{N}{p_i^{q_i}}\right],$$

where $p_i^{q_i} \leqslant N$. Taking logarithms of both sides of (0.10), we obtain

$$\begin{aligned} \log N! &= \log(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}) \\ &= \log p_1^{\alpha_1} + \log p_2^{\alpha_2} + \ldots + \log p_r^{\alpha_r} \\ &= \alpha_1 \log p_1 + \alpha_2 \log p_2 + \ldots + \alpha_r \log p_r. \end{aligned}$$

Now, we can estimate $\log N!$ in two ways. By the definition of α_i, we can see that the right hand side of the above equation has the form

$$\left(\left[\frac{N}{p_1}\right]+\left[\frac{N}{p_1^2}\right]+\left[\frac{N}{p_1^3}\right]+\ldots+\left[\frac{N}{p_1^{q_1}}\right]\right)\log p_1 \tag{0.11}$$

$$+\left(\left[\frac{N}{p_2}\right]+\left[\frac{N}{p_2^2}\right]+\left[\frac{N}{p_2^3}\right]+\ldots+\left[\frac{N}{p_2^{q_2}}\right]\right)\log p_2 \tag{0.12}$$

$$+\ldots+\left(\left[\frac{N}{p_r}\right]+\left[\frac{N}{p_r^2}\right]+\left[\frac{N}{p_r^3}\right]+\ldots+\left[\frac{N}{p_r^{q_r}}\right]\right)\log p_r. \tag{0.13}$$

We now wish to simplify the above expression by changing

$$\left[\frac{N}{p_i}\right]+\left[\frac{N}{p_i^2}\right]+\left[\frac{N}{p_i^3}\right]+\ldots+\left[\frac{N}{p_i^{q_i}}\right] \quad \text{to} \quad \frac{N}{p_i}+\frac{N}{p_i^2}+\frac{N}{p_i^3}+\ldots+\frac{N}{p_i^{q_i}},$$

where p_i remains less than N for $1 \leqslant i \leqslant r$, and with $p_i^{q_i} \leqslant N$. By removing the *integer part* notation, we introduce an error of no more than 1 per term in the estimation of $\log N!$. Indeed, by definition

$$\left[\frac{N}{p_i^{q_i}}\right] \leqslant \frac{N}{p_i^{q_i}} \leqslant \left[\frac{N}{p_i^{q_i}}\right]+1,$$

thus, for each sum of (0.11), (0.12), and (0.13), we have an error of $q_1, q_2, \ldots, q_r$ respectively, as there are exactly q_i many terms in the sum of

$$\left[\frac{N}{p_i}\right]+\left[\frac{N}{p_i^2}\right]+\left[\frac{N}{p_i^3}\right]+\ldots+\left[\frac{N}{p_i^{q_i}}\right].$$

We have now found a total error on our estimation of $\log N!$, namely

$$\begin{aligned} &q_1 \cdot \log p_1 + q_2 \cdot \log p_2 + q_3 \cdot \log p_3 + \ldots + q_r \cdot \log p_r \\ =\ &\log p_1^{q_1} + \log p_2^{q_2} + \log p_3^{q_3} + \ldots + \log p_r^{q_r} \\ \leqslant\ &\underbrace{\log N + \log N + \log N + \ldots + \log N}_{r\ times} = r \log N. \end{aligned}$$

In the above expression, we can see that r is equal to the number of primes less than or equal to N, thus by Tchebychev's Theorem,

$$r = \pi(N) < B\frac{N}{\log N},$$

where B can be taken equal to 4. We can now see that our estimate of $\log N!$ is bounded by

$$r \log N < B\frac{N}{\log N} \cdot \log N = BN.$$

This allows us to construct an upper and lower bound for $\log N!$:

$$\left(\frac{N}{p_1}+\frac{N}{p_1^2}+\frac{N}{p_1^3}+\ldots+\frac{N}{p_1^{q_1}}\right)\log p_1+\left(\frac{N}{p_2}+\frac{N}{p_2^2}+\frac{N}{p_2^3}+\ldots+\frac{N}{p_2^{q_2}}\right)\log p_2+\ldots$$
$$\ldots+\left(\frac{N}{p_r}+\frac{N}{p_r^2}+\frac{N}{p_r^3}+\ldots+\frac{N}{p_r^{q_r}}\right)\log p_r \geqslant \log N! > \left(\frac{N}{p_1}+\frac{N}{p_1^2}+\frac{N}{p_1^3}+\ldots+\frac{N}{p_1^{q_1}}\right)\log p_1$$
$$+\ldots+\left(\frac{N}{p_2}+\frac{N}{p_2^2}+\frac{N}{p_2^3}+\ldots+\frac{N}{p_2^{q_2}}\right)\log p_2+\left(\frac{N}{p_r}+\frac{N}{p_r^2}+\frac{N}{p_r^3}+\ldots+\frac{N}{p_r^{q_r}}\right)\log p_r - BN.$$

We can now modify Lemma 3.1 in the following way:

$$\frac{N}{p_i}+\frac{N}{p_i^2}+\frac{N}{p_i^3}+\ldots+\frac{N}{p_i^{q_i}}=\frac{\frac{N}{p_i}-\frac{N}{p_i^{q_i+1}}}{1-\frac{1}{p_i}}<\frac{\frac{N}{p_i}}{1-\frac{1}{p_i}}=\frac{N}{p_i-1},$$

and we also note that

$$\frac{N}{p_i}\leqslant\frac{N}{p_i}+\frac{N}{p_i^2}+\frac{N}{p_i^3}+\ldots+\frac{N}{p_i^{q_i}}.$$

Using these two facts, we find that

$$(\ast)\qquad \frac{N}{p_1-1}\log p_1+\frac{N}{p_2-1}\log p_2+\ldots+\frac{N}{p_r-1}\log p_r\geqslant\log N!$$
$$>\frac{N}{p_1}\log p_1+\frac{N}{p_2}\log p_2+\ldots+\frac{N}{p_r}\log p_r-BN.$$

We wish to simplify the above inequality even further, so as it has the following form:

$$(\ast\ast)\qquad N\left(\frac{\log p_1}{p_1}+\frac{\log p_2}{p_2}+\ldots+\frac{\log p_r}{p_r}+K\right)>\log N!$$
$$>N\left(\frac{\log p_1}{p_1}+\frac{\log p_2}{p_2}+\ldots+\frac{\log p_r}{p_r}-B\right),$$

where K is a positive constant. To do this, we must find the difference between the left and right hand sides of $(\ast)$ and $(\ast\ast)$. We can begin with the obvious case, which is the right side. Clearly, N has simply been factored from all the terms on the right hand side of $(\ast)$, thus the right hand sides of $(\ast)$ and $(\ast\ast)$ are equivalent. For the former case, we first find how much the left hand side of $(\ast)$ differs from

$$N\left(\frac{\log p_1}{p_1}+\frac{\log p_2}{p_2}+\ldots+\frac{\log p_r}{p_r}\right). \qquad (0.14)$$

This difference will ultimately be our K value. After factoring out N from $(\ast)$, we find that the difference for each term is equal to

$$\frac{\log p_i}{p_i-1}-\frac{\log p_i}{p_i}=\frac{p_i\log p_i-(p_i-1)\log p_i}{p_i(p_i-1)}=\frac{\log p_i}{p_i(p_i-1)},$$

for $i = 1, 2, 3, \ldots, r$, thus the total difference between the left hand side of ($\circledast$) and (0.14) is

$$N\left(\frac{\log p_1}{p_1(p_1-1)} + \frac{\log p_2}{p_2(p_2-1)} + \ldots + \frac{\log p_r}{p_r(p_r-1)}\right). \tag{0.15}$$

We now claim that for any $a \geqslant 2$

$$\frac{\log a}{a(a-1)} < \frac{1}{a\sqrt{a}}.$$

It is clear that the above inequality implies $\log a < \frac{a-1}{\sqrt{a}}$, which is equivalent to $2\log a < 2\left(\frac{a-1}{\sqrt{a}}\right)$. Working with the right hand side of the inequality, we find that for $a \geqslant 2$,

$$2\left(\frac{a-1}{\sqrt{a}}\right) = 2\left(\frac{\sqrt{a}(a-1)}{a}\right) = 2\left(\sqrt{a} - \frac{\sqrt{a}}{a}\right) \geqslant 2\left(\sqrt{a} - \frac{\sqrt{a}}{2}\right) = \sqrt{a}.$$

Now, using Lemma 3.4, since

$$2\log a < \sqrt{a} \quad \text{ and } \quad 2\left(\frac{\mathrm{a}-1}{\sqrt{\mathrm{a}}}\right) \geqslant \sqrt{\mathrm{a}},$$

we can therefore conclude that $2\log a < 2\left(\frac{a-1}{\sqrt{a}}\right)$.

We can now find an upper bound for the sum inside the brackets of (0.15):

$$\begin{aligned}\frac{\log p_1}{p_1(p_1-1)} + \frac{\log p_2}{p_2(p_2-1)} + \ldots + \frac{\log p_r}{p_r(p_r-1)} \\ < \frac{1}{p_1\sqrt{p_1}} + \frac{1}{p_2\sqrt{p_2}} + \ldots + \frac{1}{p_r\sqrt{p_r}} \\ < 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots + \frac{1}{N\sqrt{N}}.\end{aligned}$$

Using Lemma 3.2, taking $s = \frac{3}{2}$, we find that the last sum of our previous inequality is bounded:

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots + \frac{1}{N\sqrt{N}} < \frac{\frac{3}{2}}{\frac{3}{2}-1} = 3.$$

We can therefore conclude that the difference between the left hand side of ($\circledast$) and (0.14), namely K, can be taken equal to 3.

We now find another estimate for $\log N!$ using Lemma 3.3. Recall that

$$\Phi_1 \cdot \sqrt{N} \cdot \left(\frac{N}{e}\right)^N < N! < \Phi_2 \cdot \sqrt{N} \cdot \left(\frac{N}{e}\right)^N.$$

where $\Phi_1 = e\sqrt{\frac{4}{5}}$ and $\Phi_2 = e$. Taking logarithms of each part of the inequality, we obtain

$$\log\left(\Phi_1 \cdot \sqrt{N} \cdot \left(\frac{N}{e}\right)^N\right) < \log N! < \log\left(\Phi_2 \cdot \sqrt{N} \cdot \left(\frac{N}{e}\right)^N\right)$$

$$\log \Phi_1 + \log\sqrt{N} + \log\left(\frac{N}{e}\right)^N < \log N! < \log \Phi_2 + \log\sqrt{N} + \log\left(\frac{N}{e}\right)^N$$

$$(\S)\ \log \Phi_1 + \frac{1}{2}\log N + N(\log N - \log e) < \log N! < \log \Phi_2 + \frac{1}{2}\log N + N(\log N - \log e).$$

We proved previously that $\frac{n}{\log n} < \frac{x}{\log x}$ for $n \geqslant 3$ and $n < x$. Rearranging this inequality, we find that

$$\frac{\log x}{x} < \frac{\log n}{n}.$$

Thus, it is clear that for $N \geqslant 3$,

$$\frac{1}{2}\frac{\log N}{N} < \frac{1}{2}\frac{\log 3}{3} < 0.08$$

and

$$\frac{\log \Phi_2}{N} < \frac{\log e}{3} < 0.15.$$

Using these two inequalities, we can modify (§) in the following way

$$\log \Phi_1 + \frac{1}{2}\log N + N(\log N - \log e) < \log N! < \log \Phi_2 + \frac{1}{2}\log N + N(\log N - \log e)$$

$$N(\log N - \log e) < \log N! < N\frac{\log \Phi_2}{N} + N\frac{\log N}{2N} + N(\log N - \log e)$$

$$N(\log N - \log e) < \log N! < N\left(\frac{\log \Phi_2}{N} + \frac{1}{2}\frac{\log N}{N} + \log N - \log e\right)$$

$$N(\log N - \log e) < \log N! < N\left(0.15 + 0.08 + \log N - \log e\right)$$

$$N(\log N - \log e) < \log N! < N\left(\log N - \log e + 0.23\right).$$

Now, we recall from (∗∗) that

$$N\left(\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} + K\right) > \log N!$$
$$> N\left(\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} - B\right).$$

Comparing this inequality with the previous inequality, we come to two conclusions. First,

$$\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} + K > \log N - \log e,$$

and second,

$$\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} - B < \log N - \log e + 0.23.$$

Rearranging the two inequalities, we obtain

$$\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} - \log N > -K - \log e$$

and

$$\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} - \log N < B - \log e + 0.23.$$

Using the fact that K and B can be taken equal to 3 and 4 respectively, we can calculate the right hand sides of the two expressions above. Thus, $-K - \log e = -3 - 0.43429 = -3.43429$ and $B - \log e + 0.23 = 4 - 0.43429 + 0.23 = 3.79571$. Taking the larger of the absolute value of these two numbers, namely $|-3.43429| < |3.79571| < 4$, we obtain

$$-4 < \frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} - \log N < 4.$$

This was the result we were looking for, which therefore completes this proof.

□

CHAPTER 4

Mertens' Second Theorem

We begin this chapter by introducing an extremely useful formula which, with the help of Mertens' First Theorem, will enable us to prove Mertens' Second Theorem.

THEOREM 4.1. (Abel's Formula) *Consider the sum*

$$S = a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n,$$

where $a_1, a_2, a_3, \ldots, a_n$ *and* $b_1, b_2, b_3, \ldots, b_n$ *are any two sequences of numbers, and denote the sums* $b_1,\ b_1 + b_2,\ b_1 + b_2 + b_3,\ \ldots,\ b_1 + b_2 + b_3 + \ldots + b_n$ *by* $B_1,\ B_2,\ B_3,\ \ldots,\ B_n$ *respectively. Then*

$$S = (a_1 - a_2)B_1 + (a_2 - a_3)B_2 + (a_3 - a_4)B_3 + \ldots + (a_{n-1} - a_n)B_{n-1} + a_nB_n.$$

THEOREM 4.2. (Mertens' Second Theorem) *Given any* $N > 1$, *for all* $p < N$ *with* p *prime, the expression*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{p} - \ln\ln N \tag{0.16}$$

is less than some relatively small constant T, *where* T *is independent of* T.

PROOF. To begin this proof, we first examine the series

$$S = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r},$$

where $p_1,\ p_2,\ \ldots,\ p_r$ are all the primes less than or equal to any integer N. We now set

$$a_1 = \frac{1}{\log p_1},\ a_2 = \frac{1}{\log p_2},\ a_3 = \frac{1}{\log p_3}, \ldots,\ a_r = \frac{1}{\log p_r}$$

$$b_1 = \frac{\log p_1}{p_1},\ b_2 = \frac{\log p_2}{p_2},\ b_3 = \frac{\log p_3}{p_3}, \ldots,\ b_r = \frac{\log p_r}{p_r}.$$

Using Theorem 4.1, denoting the sums b_1, $b_1 + b_2$, $b_1 + b_2 + b_3$, $\ldots$, $b_1 + b_2 + b_3 + \ldots + b_r$ by B_1, B_2, B_3, $\ldots$, B_r respectively, we obtain

$$\begin{aligned} B_1 &= \frac{\log p_1}{p_1}, \\ B_2 &= \frac{\log p_1}{p_1} + \frac{\log p_2}{p_2}, \\ &\vdots \\ B_r &= \frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r}. \end{aligned}$$

We can now write our series S in terms of $a_1, a_2, \ldots, a_r$, and $b_1, b_2, \ldots, b_r$. Clearly

$$\begin{aligned} &a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots + a_r b_r \\ &= \frac{1}{\log p_1} \cdot \frac{\log p_1}{p_1} + \frac{1}{\log p_2} \cdot \frac{\log p_2}{p_2} + \ldots + \frac{1}{\log p_r} \cdot \frac{\log p_r}{p_r} \\ &= \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r}, \end{aligned}$$

but by Theorem 4.1,

$$\begin{aligned} &a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots + a_r b_r \\ &= (a_1 - a_2) B_1 + (a_2 - a_3) B_2 + (a_3 - a_4) B_3 + \ldots + (a_{r-1} - a_r) B_{r-1} + a_r B_r, \end{aligned}$$

thus

$$\begin{aligned} &\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r} \\ &= \left(\frac{1}{\log p_1} - \frac{1}{\log p_2} \right) B_1 + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3} \right) B_2 \\ &+ \left(\frac{1}{\log p_3} - \frac{1}{\log p_4} \right) B_3 + \ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r} \right) B_{r-1} + \frac{1}{\log p_r} \cdot B_r \\ (*) \quad &= \left(\frac{1}{\log p_1} - \frac{1}{\log p_2} \right) B_1 + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3} \right) B_2 \\ &+ \left(\frac{1}{\log p_3} - \frac{1}{\log p_4} \right) B_3 + \ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r} \right) B_{r-1} \\ &+ \left(\frac{1}{\log p_r} - \frac{1}{\log N} \right) B_r + \frac{1}{\log N} \cdot B_r. \end{aligned}$$

Now by Theorem 3.5, we can deduce that:

$$\begin{aligned}
&\left|\frac{\log p_1}{p_1} - \log p_1\right| < K \Rightarrow |B_1 - \log p_1| < K \Rightarrow B_1 < \log p_1 + K,\\
&\left|\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} - \log p_2\right| < K \Rightarrow |B_2 - \log p_2| < K \Rightarrow B_2 < \log p_2 + K,\\
&\left|\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \frac{\log p_3}{p_3} - \log p_3\right| < K \Rightarrow B_3 < \log p_3 + K,\\
&\qquad\qquad\vdots\\
&\left|\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} - \log p_r\right| < K \Rightarrow B_r < \log p_r + K,\\
&\left|\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_r}{p_r} - \log N\right| < K \Rightarrow B_r < \log N + K,
\end{aligned}$$

where K can be taken equal to 4. Thus we can conclude that

$$\begin{aligned}
&B_1 < \log p_1 + K, \quad B_2 < \log p_2 + K, \quad B_3 < \log p_3 + K, \ldots\\
&\ldots, \quad B_r < \log p_r + K, \quad \text{and} \quad B_r < \log N + K.
\end{aligned}$$

From above, it follows that

$$\begin{aligned}
&\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r}\\
&< \left(\frac{1}{\log p_1} - \frac{1}{\log p_2}\right)(\log p_1 + K) + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3}\right)(\log p_2 + K)\\
&+ \left(\frac{1}{\log p_3} - \frac{1}{\log p_4}\right)(\log p_3 + K) + \ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r}\right)\log p_{r-1} + K\\
&+ \left(\frac{1}{\log p_r} - \frac{1}{\log N}\right)(\log p_r + K) + \frac{1}{\log N}\cdot(\log N + K)\\
&= \left(\frac{1}{\log p_1} - \frac{1}{\log p_2}\right)\log p_1 + \frac{K}{\log p_1} - \frac{K}{\log p_2} + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3}\right)\log p_2 + \frac{K}{\log p_2} - \frac{K}{\log p_3}\\
&+ \left(\frac{1}{\log p_3} - \frac{1}{\log p_4}\right)\log p_3 + \frac{K}{\log p_3} - \frac{K}{\log p_4} + \ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r}\right)\log p_{r-1}\\
&+ \frac{K}{\log p_{r-1}} - \frac{K}{\log p_r} + \left(\frac{1}{\log p_r} - \frac{1}{\log N}\right)\log p_r + \frac{K}{\log p_r} - \frac{K}{\log N} + \frac{1}{\log N}\log N + \frac{K}{\log N}.
\end{aligned}$$

The bound on our series $S = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r}$ can simplify to

$$1 + \frac{-\log p_1}{\log p_2} + \frac{K}{\log p_1} + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3}\right)\log p_2 + \left(\frac{1}{\log p_3} - \frac{1}{\log p_4}\right)\log p_3$$
$$+ \ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r}\right)\log p_{r-1} + \left(\frac{1}{\log p_r} - \frac{1}{\log N}\right)\log p_r + \frac{1}{\log N}\log N,$$

which after further rearrangement yields

$$(**) \quad 1 + \left[\frac{-\log p_1}{\log p_2} + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3}\right)\log p_2 + \left(\frac{1}{\log p_3} - \frac{1}{\log p_4}\right)\log p_3 + \ldots\right.$$
$$\left. + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r}\right)\log p_{r-1} + \left(\frac{1}{\log p_r} - \frac{1}{\log N}\right)\log p_r + \frac{1}{\log N}\log N\right] + \frac{K}{\log p_1}.$$

Now we will focus on the expression found inside the square brackets. We can expand the expression to look like

$$\frac{-\log p_1}{\log p_2} + \frac{\log p_2}{\log p_2} - \frac{\log p_2}{\log p_3} + \frac{\log p_3}{\log p_3} - \frac{\log p_3}{\log p_4} + \ldots$$
$$\ldots + \frac{\log p_{r-1}}{\log p_{r-1}} - \frac{\log p_{r-1}}{\log p_r} + \frac{\log p_r}{\log p_r} - \frac{\log p_r}{\log N} + \frac{\log N}{\log N},$$

then factor the common denominators, which will yield the following

$$(\log p_2 - \log p_1)\frac{1}{\log p_2} + (\log p_3 - \log p_2)\frac{1}{\log p_3} + (\log p_4 - \log p_3)\frac{1}{\log p_4}$$
$$+ \ldots + (\log p_r - \log p_{r-1})\frac{1}{\log p_r} + (\log N - \log p_r)\frac{1}{\log N}.$$

We can now estimate the above expression geometrically by constructing a graph to model the series. We can depict the expression by the sum of $N-1$ rectangles. The area of the first rectangle will have a base of length $(\log p_2 - \log p_1)$ and a height of length $\frac{1}{\log p_2}$, as per the first term in our expression. The second rectangle will have a base of length $(\log p_3 - \log p_2)$, and a height of length $\frac{1}{\log p_3}$. This pattern continues up to our $(N-1)^{th}$ rectangle, which will have a base of length $(\log N - \log p_r)$ and a height of length $\frac{1}{\log N}$. These rectangles are plotted in Figure (4.1). We now observe that when $x = \log p_2$, that $y = \frac{1}{\log p_2}$. Similarly, when $x = \log p_3$, $y = \frac{1}{\log p_3}$, and so on, up to $x = \log N$, and $y = \frac{1}{\log N}$. In general we find that $y = \frac{1}{x}$, which is a hyperbola that intersects the top right corner of each rectangle. This hyperbola is also shown in Figure 4.1.

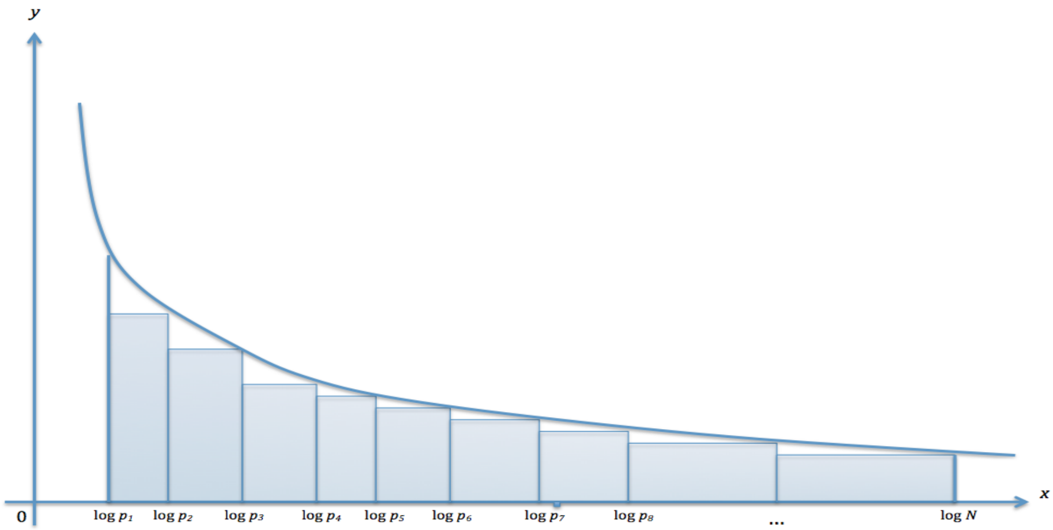

Figure 4.1

Now that we have defined the hyperbola such that the shaded rectangles are all below its graph, we have thus found an upper bound on all the rectangles, and therefore an upper bound on the series S. We simply need to calculate the area below the hyperbola $y = \frac{1}{x}$ between the points $x = \log p_1$ and $x = \log N$. This is done easily using calculus. Clearly

$$(0.17) \qquad \int_{\log p_1}^{\log N} \frac{1}{x} dx = (\ln x)\Big|_{\log p_1}^{\log N} = \ln(\log N) - \ln(\log p_1).$$

We can therefore conclude that the expression in the square brackets from $(**)$, is bounded by $\ln(\log N) - \ln(\log p_1)$, and thus

$$(0.18) \qquad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r} < 1 + \ln(\log N) - \ln(\log p_1) + \frac{K}{\log p_1}.$$

We now focus our attention on finding a lower bound for our series $S = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r}$. Recall that

$$B_i = \frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_i}{p_i},$$

where $i = (1, 2, \ldots, r)$, and $p_r \leqslant N$. Adding $\frac{\log p_{i+1}}{p_{i+1}}$ and subtracting a, where $a > \frac{\log p_{i+1}}{p_{i+1}}$, we find that

$$(0.19) \qquad B_i > \frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \ldots + \frac{\log p_i}{p_i} + \frac{\log p_{i+1}}{p_{i+1}} - a.$$

Here, we can let $a = \frac{\log 3}{3}$, as for $n \geqslant 3$, $\frac{\log 3}{3} < \frac{\log n}{n}$. Now, as a result of Mertens' First Theorem, we find that

$$-K < \frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \frac{\log p_3}{p_3} + \ldots + \frac{\log p_i}{p_i} + \frac{\log p_{i+1}}{p_{i+1}} - \log p_{i+1},$$

which implies

$$\log p_{i+1} - K < \frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \frac{\log p_3}{p_3} + \ldots + \frac{\log p_i}{p_i} + \frac{\log p_{i+1}}{p_{i+1}}.$$

Returning to equation (0.19), we can conclude that

$$\begin{aligned} B_i &> \left(\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \frac{\log p_3}{p_3} + \ldots + \frac{\log p_i}{p_i} + \frac{\log p_{i+1}}{p_{i+1}}\right) - a \\ &> (\log p_{i+1} - K) - a, \end{aligned}$$

for $1 \leqslant i < r$. Using the same argument, we find that

$$\frac{\log p_1}{p_1} + \frac{\log p_2}{p_2} + \frac{\log p_3}{p_3} + \ldots + \frac{\log p_r}{p_r} > \log N - K - a,$$

with the $-a$ added in for convenience. From above, we can conclude that

$$\begin{aligned} &B_1 > \log p_2 - K - a, \quad B_2 > \log p_3 - K - a, \ldots \\ &\ldots, \quad B_{r-1} > \log p_r - K - a, \quad B_r > \log N - K - a. \end{aligned}$$

Now, referring back to $(*)$, we find that

$$\begin{aligned} &\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_n} \\ &> \left(\frac{1}{\log p_1} - \frac{1}{\log p_2}\right)(\log p_2 - K - a) + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3}\right)(\log p_3 - K - a) \\ &+ \left(\frac{1}{\log p_3} - \frac{1}{\log p_4}\right)(\log p_4 - K - a) + \ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r}\right)(\log p_r - K - a) \\ &+ \left(\frac{1}{\log p_r} - \frac{1}{\log N}\right)(\log N - K - a) + \frac{1}{\log N} \cdot (\log N - K - a) \\ &= \left(\frac{1}{\log p_1} - \frac{1}{\log p_2}\right)\log p_2 - \frac{K+a}{\log p_1} + \frac{K+a}{\log p_2} + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3}\right)\log p_3 \\ &- \frac{K+a}{\log p_2} + \frac{K+a}{\log p_3} + \left(\frac{1}{\log p_3} - \frac{1}{\log p_4}\right)\log p_4 - \frac{K+a}{\log p_3} + \frac{K+a}{\log p_4} + \ldots \\ &\ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r}\right)\log p_r - \frac{K+a}{\log p_{r-1}} + \frac{K+a}{\log p_r} + \left(\frac{1}{\log p_r} - \frac{1}{\log N}\right)\log N \\ &- \frac{K+a}{\log p_r} + \frac{K+a}{\log N} + \frac{1}{\log N}\log N - \frac{K+a}{\log N}. \end{aligned}$$

With further rearrangement, we obtain

$$= \left(\frac{1}{\log p_1} - \frac{1}{\log p_2}\right)\log p_2 - \frac{K+a}{\log p_1} + \left(\frac{1}{\log p_2} - \frac{1}{\log p_3}\right)\log p_3$$
$$+ \left(\frac{1}{\log p_3} - \frac{1}{\log p_4}\right)\log p_4 + \ldots + \left(\frac{1}{\log p_{r-1}} - \frac{1}{\log p_r}\right)\log p_r$$
$$+ \left(\frac{1}{\log p_r} - \frac{1}{\log N}\right)\log N + \frac{1}{\log N}\log N$$
$$= \frac{\log p_2}{\log p_1} - \frac{\log p_2}{\log p_2} + \frac{\log p_3}{\log p_2} - \frac{\log p_3}{\log p_3} + \frac{\log p_4}{\log p_3} - \frac{\log p_4}{\log p_4} + \ldots + \frac{\log p_r}{\log p_{r-1}} - \frac{\log p_r}{\log p_r}$$
$$+ \frac{\log N}{\log p_r} - \frac{\log N}{\log N} + \frac{1}{\log N}\log N - \frac{K+a}{\log p_1}$$
$$= \left[(\log p_2 - \log p_1)\frac{1}{\log p_1} + (\log p_3 - \log p_2)\frac{1}{\log p_2} + (\log p_4 - \log p_3)\frac{1}{\log p_3} + \ldots\right.$$
$$\left.\ldots + (\log N - \log p_r)\frac{1}{\log p_r}\right] + 1 - \frac{K+a}{\log p_1}.$$

In this last step, we omitted the $-\frac{\log N}{\log N}$ term, and replaced it with $-\frac{\log p_1}{\log p_1}$. Now, as we have previously done with the upper bound on S, we can see that the sum inside the square brackets above is simply the sum of differently sized rectangles. This time, the first rectangle will have a base of length $(\log p_2 - \log p_1)$ and a height of length $\frac{1}{\log p_1}$. The second rectangle will have a base of length $(\log p_3 - \log p_2)$ and a height of length $\frac{1}{\log p_2}$, and so on, up to a rectangle with a base of length $(\log N - \log p_r)$ and height of length $\frac{1}{\log p_r}$. These rectangles are shown in Figure 4.2.

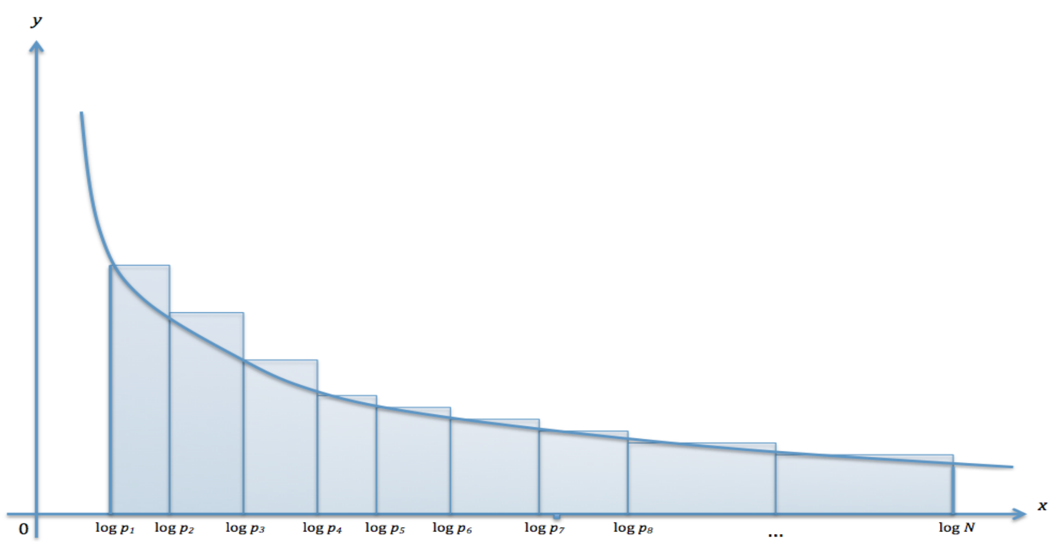

Figure 4.2

We can see that when $x = \log p_1$, $y = \frac{1}{\log p_1}$. This pattern holds for all p_r, with $r \leqslant N$. We also notice that, as before, these coordinates define the function $y = \frac{1}{x}$, only this time, the curve intersects the top left hand corner of the rectangles, as shown in Figure 4.2. This implies that the area bounded by the curve between $\log p_1$ and $\log N$ is less than the area of the shaded rectangles. We simply need to calculate this area, which will in turn give us a lower bound for our series, but we have already calculated this to be $\ln(\log N) - \ln(\log p_1)$ from (0.17).

Now that we have a lower bound for the sum of the rectangles, we find that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r} > \ln(\log N) - \ln(\log p_1) + 1 - \frac{K+a}{\log p_1}. \tag{0.20}$$

Next, combining equations (0.18) and (0.20), we find that for $S = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r}$

$$\ln(\log N) - \ln(\log p_1) + 1 - \frac{K+a}{\log p_1} < S < 1 + \ln(\log N) - \ln(\log p_1) + \frac{K}{\log p_1}. \tag{0.21}$$

Now, as opposed to using logarithms of two different bases in our expression, we will use the fact that

$$\log N = M \ln N,$$

where $M = \log e$. This changes (0.21) to

$$\ln(M \ln N) - \ln(\log p_1) + 1 - \frac{K+a}{\log p_1} < S < 1 + \ln(M \ln N) - \ln(\log p_1) + \frac{K}{\log p_1}$$

$$\ln \ln N + \ln M - \ln(\log p_1) + 1 - \frac{K+a}{\log p_1} < S < 1 + \ln \ln N + \ln M - \ln(\log p_1) + \frac{K}{\log p_1}.$$

From here we can simply evaluate our expression using the fact that $M = \log e = 0.43429...$, $p_1 = 2$, $K = 4$, and $a = \frac{\log 3}{3} = 0.15904....$ Clearly,

$$\begin{aligned}
&\ln \ln N + \ln M - \ln(\log p_1) + 1 - \frac{K+a}{\log p_1} \\
&= \ln \ln N + \ln(0.43429) - \ln(\log 2) + 1 - \frac{4 + 0.15904}{\log 2} \\
&= \ln \ln N + (-0.83404) - (-1.20054) + 1 - (13.81603) \\
&= \ln \ln N - 12.44953 > \ln \ln N - 13.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&1 + \ln \ln N + \ln M - \ln(\log p_1) + \frac{K}{\log p_1} \\
&= \ln \ln N + 1 + (-0.83404) - (-1.20054) + \frac{4}{0.30102} \\
&= \ln \ln N + 14.65465 < \ln \ln N + 15.
\end{aligned}$$

Comparing the two inequalities above, we finally obtain

$$\ln \ln N - T < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots + \frac{1}{p_r} < \ln \ln N + T,$$

where $T = 15$, the larger of the upper and lower bounds. This completes the proof.

□

CHAPTER 5

Concluding Remarks

This project explored how prime numbers are distributed throughout the set of real numbers. We proved that there exists an upper and lower bound on $\pi(N)$, which is dependent on N. It is fascinating to see how such a complicated yet relatively elementary proof can yield such an elegant result. We used this result to explore further problems in the study of prime numbers, specifically, Mertens' First and Second Theorem.

In Mertens' First Theorem, we used similar arguments as in the proof of Tchebychev's Theorem, and simply found the magnitude of errors between different series. The proof of Mertens' Second Theorem was more geometrically intuitive. We used Mertens' First Theorem as well as Abel's Formula to transform our series into another more convenient series, which could be interpreted geometrically. This allowed us to find an upper and lower bound using integration, which enabled us to ultimately complete the proof.

From this point, the next step in research could be exploring the proof the the Prime Number Theorem. This would involve a much more complicated proof, using methods from Complex Analysis.

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