Multiple Choice Questions.

1. C.
2. C.
3. E.
4. D.
5. C.
6. A.
7. B.
8. B.
9. C.
10. B.
11. A.
12. D.
13. E.
14. D.
15. D.
16. B.
17. B.
18. E.
19. A.
20. C.

Full Solution.

1. Letting $c = y - x$, we are looking for the largest and smallest values of c, subject to $x^2 + y^2 = 1$. As $y = c + x$, this means $x^2 + (c+x)^2 = 1$. That is, $2x^2 + 2cx + c^2 - 1 = 0$. By the quadratic equation, $x = \frac{-2c \pm \sqrt{(2c)^2 - 4(2)(c^2-1)}}{2(2)}$. This is a real number if and only if $(2c)^2 - 4(2)(c^2 - 1) \geq 0$; that is, if and only if $-4c^2 + 8 \geq 0$. But this means that $c^2 \leq 2$. Therefore, the largest possible value of c is $a = \sqrt{2}$, and the smallest is $b = -\sqrt{2}$. The answer is $\sqrt{2} - (-\sqrt{2}) = 2\sqrt{2}$.
2. Every positive integer can be written as $3k$, $3k+1$ or $3k+2$, for some integer k. If $n = 3k$, then $\lfloor n^2/3 \rfloor = \lfloor 3k^2 \rfloor = 3k^2$. This cannot possibly be prime unless $k = 1$. Indeed, $k = 1$ does work, and $n = 3$ is a valid solution. If $n = 3k+1$, then $\lfloor n^2/3 \rfloor = \lfloor \frac{9k^2+6k+1}{3} \rfloor = \lfloor 3k^2 + 2k + \frac{1}{3} \rfloor = 3k^2 + 2k = k(3k+2)$. This cannot be prime unless $k = 1$, and again, $k = 1$ does work, since $n = 4$ is a valid solution. Finally, suppose that $n = 3k + 2$. Then $\lfloor n^2/3 \rfloor = \lfloor \frac{9k^2+12k+4}{3} \rfloor = \lfloor 3k^2 + 4k + 1 + \frac{1}{3} \rfloor = 3k^2 + 4k + 1 = (3k+1)(k+1)$. This is clearly not prime unless $k = 0$, but even then, we get 1, which is not prime. Therefore, the only solutions are $n = 3$ and 4.