TD Canada Trust

Lakehead UNIVERSITY

Department of Mathematical Sciences

April 25, 2012 HIGH SCHOOL MATHEMATICS COMPETITION

SENIOR COMPETITION

Grades 11 and 12

Name: ______________________________

E-Mail: ______________________________

School & Grade: ______________________________

Telephone: ______________________________

Question #	*Your Answer*	For Markers Use only
1		/4
2		/4
3		/4
4		/4
5		/4
6		/4
7		/4
8		/4
9		/4
10		/4
11		/4
12		/4
13		/4
14		/4
15		/4
16		/4
17		/4
18		/4
19		/4
20		/4
	Number of Unanswered Questions	x 1
		/80

Full Solution For Markers use (full solution):	
Question #	Mark
1	/10
2	/10
Full Solution Total	/20

Instructions for full solution questions:

- Place your solutions to these questions in this answer booklet.
- If you require additional space, use the back of the page but leave a note indicating this to the marker.

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Multiple Choice (80 Marks)

Place all answers in the multiple choice boxes on the front page of the answer booklet.

Questions 1-20 below are worth:

4 marks for a correct answer
1 mark for a blank answer
0 marks for an incorrect answer.

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(1) Suppose that $\log a = 2 + \log b$. Then $a/b =$

(A) $\log 2$ (B) 2 (C) 100 (D) 1024 (E) impossible to determine

(2) We have a group of 60 students, each of whom is either wearing running shoes or sandals. Given that (i) there are 12 boys wearing running shoes, (ii) there are 34 girls and (iii) there are 27 people wearing sandals, the number of girls wearing sandals is

(A) 0 (B) 12 (C) 13 (D) 14 (E) 21

(3) The sequence $1, x, 9, y, \ldots$ is an arithmetic progression. The product xy is equal to

(A) 13 (B) 18 (C) 45 (D) 58 (E) 65

(4) Suppose that $\sin(x) = 2/3$. Then $\tan^2(x) + \frac{1}{1+\cos^2(x)}$ is equal to

(A) $9/14$ (B) $4/5$ (C) $56/45$ (D) $101/70$ (E) $\cos(x)$

(5) For how many integers n is $\frac{4n+28}{n-2}$ an integer?

(A) 1 (B) 12 (C) 18 (D) 24 (E) infinitely many

(6) How many polynomials $f(x)$ are there with integer coefficients such that $f(2) = 4$ and $f(4) = 5$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinitely many

(7) Recall that an integer $p > 1$ is said to be prime if its only positive integer divisors are 1 and p. Let n be the smallest positive integer which is simultaneously (i) the sum of 3 distinct primes (ii) the product of 2 distinct primes and (iii) the sum of 3 distinct positive perfect squares. Which of the following statements is true?

(A) $n < 10$ (B) $10 \le n < 20$ (C) $20 \le n < 40$ (D) $40 \le n < 80$ (E) $n \ge 80$

(8) For how many positive integers n is $n^4 + n^2 + 1$ prime?

(A) 0 (B) 1 (C) 2 (D) 4 (E) infinitely many

(9) Which of the following is an integer?

(A) $2^{23}/17$ (B) $\sqrt{1000}$ (C) $\frac{2}{1+\sqrt{3}} + \frac{2}{1-\sqrt{3}}$ (D) $\sqrt{10^{40} - 1}$ (E) none of these

(10) Let $s_1 = 12$. For each $n \ge 2$, if s_{n-1} is even, then $s_n = \frac{s_{n-1}}{2}$. If s_{n-1} is odd, then $s_n = 3s_{n-1} + 1$. Then s_{2012} is

(A) 1 (B) 2 (C) 4 (D) 6 (E) 8

(11) If n is a positive integer, write $n! = n(n-1)(n-2)\cdots(2)(1)$. For example, $5! = 5(4)(3)(2)(1) = 120$. Let k be the sum of the distinct primes dividing $25!$. Which of the following is true?

(A) $k = 10^2$ (B) $10^2 < k \leq 10^3$ (C) $10^3 < k \leq 10^5$ (D) $10^5 < k \leq 10^8$ (E) $k > 10^8$

(12) The three numbers $\sqrt{5}$, $\sqrt[3]{5}$ and $\sqrt[6]{5}$ are consecutive terms in a geometric progression. The next term is

(A) $\sqrt[18]{5}$ (B) $\sqrt[9]{5}$ (C) $\sqrt[10]{5}$ (D) 1 (E) π

(13) If x is an angle in radians, how many real numbers x, with $0 < x < \pi/2$, satisfy $\log(\tan(x)) + \log(\cot(x)) = 0$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) infinitely many

(14) If we let $a \circ b = \frac{a-b}{b}$, then $6 \circ (3 \circ 2) =$

(A) $-1/2$ (B) 0 (C) $1/2$ (D) 11 (E) 36

(15) The polynomial $x^3 + \alpha x + \beta$ has three distinct real roots, a, b and c. Calculating $a^2 + b^2 + c^2$, we obtain

(A) 2 (B) -2α (C) α (D) β (E) 12

(16) If real numbers x and y satisfy $2x^2 + y^2 = 6x$, then the largest possible value of $x^2 + y^2 + 2x$ is

(A) 14 (B) 15 (C) 16 (D) 17 (E) none of these

(17) Violet, Wilbur, Xena, Yolanda and Zeke are considering going to a party. If Violet goes, then Wilbur will go too. Only one of Wilbur or Xena will go. Xena and Yolanda will either both go or both not go. At least one of Yolanda and Zeke will go. If Zeke goes, then both Violet and Yolanda will go. How many of these five people go to the party?

(A) 1 (B) 2 (C) 3 (D) 4 (E) this is impossible

(18) Expanding $(1+x)^{2012}$, we obtain $a_0 + a_1 x + a_2 x^2 + \cdots + a_{2012} x^{2012}$. What is $a_0 + a_2 + a_4 + a_6 + \cdots + a_{2012}$?

(A) 2012 (B) 2^{1006} (C) $2^{1006} - 1$ (D) 2^{2012} (E) 2^{2011}

(19) Consider a regular 16-gon. How many different rectangles can we obtain by using 4 vertices from the 16-gon?

(A) 28 (B) 30 (C) 224 (D) 240 (E) 1820

(20) What is the largest integer less than or equal to $(\sqrt{2} + \sqrt{3})^6$?

(A) 967 (B) 968 (C) 969 (D) 970 (E) 971

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Full Solutions (20 Marks)

Place your solutions to these questions in the space provided. Each question is worth 10 marks.

You must show sufficient work to receive full marks, but if you do not completely answer a question you may still receive partial marks for showing work. So **show your work!**

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1. Given that x and y are real numbers with $x^2 + y^2 = 1$, let a be the largest possible value of $y - x$ and b the smallest possible value of $y - x$. Compute $a - b$, and explain your reasoning.

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2. For any real number x, write $\lfloor x \rfloor$ for the largest integer less than or equal to x. (For instance, $\lfloor 3.8 \rfloor = 3$.) Find all positive integers n such that $\lfloor \frac{n^2}{3} \rfloor$ is prime. Be sure to explain why no other values of n will work.