# A Glimpse into Topology

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## Abstract

This report covers the basics of Topology, including topological spaces, density, continuity, compactness and nets. It is followed by a short discussion of a few topics in metric space theory, such as metrizable spaces and completeness. Finally, the concepts of Topology are applied to metric spaces to derive three interesting results: Cantor's Intersection Theorem, Bourbaki's Mittag-Leffler Theorem and Baire's Theorem.

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# Contents

CHAPTER 1

# Introduction

Topology was first introduced by German mathematician Felix Hausdorff in 1914. It is a theory that is applied to several branches of mathematics, including Algebra and Analysis.

The goal of this project was to study the basics of Topology and its applications in Analysis, including popular results like Baire's Theorem.

CHAPTER 2

# Topology

In this chapter, we will discuss basic concepts in Topology.

## 1. Topological Spaces

We begin with definitions of topology and other related terms.

DEFINITION 2.1. Let $\tau$ be a collection of subsets of a set $X$. Then $\tau$ is a *topology* on $X$ if,

(1) for any $A \subset \tau$, $\bigcup A \in \tau$,
(2) for any $A_1, A_2 \in \tau$, $A_1 \cap A_2 \in \tau$,
(3) $\phi \in \tau$,
(4) $X \in \tau$.

EXAMPLE 2.2. If $X$ is the set of colours, then the following is a topology on it:

$$\tau = \{\{red\}, \{green\}, \{red, green\}, \phi, C\}$$

DEFINITION 2.3. A tuple $(X, \tau)$ is a *topological space* if $\tau$ is a topology on $X$.

DEFINITION 2.4. Let $(X, \tau)$ be a topological space. A subset $U$ of $X$ is *open* if $U \in \tau$.

DEFINITION 2.5. Let $(X, \tau)$ be a topological space. A subset $F$ of $X$ is *closed* if $X - F \in \tau$.

## 2. Closure

DEFINITION 2.6. Let $(X, \tau)$ be a topological space and $A \subset X$. Then the *closure* of $A$, denoted as $\overline{A}$, is a closed subset of $X$ with the following properties:

(1) $A \subset \overline{A}$.
(2) If $F$ is a closed subset of $X$ and $A \subset F$, then $\overline{A} \subset F$.

THEOREM 2.7. *Let $(X, \tau)$ be a topological space and $A, B \subset X$. If $A \subset B$, then $\overline{A} \subset \overline{B}$.*

PROOF. If $A \subset B$, then $A \subset \overline{B}$. Therefore $\overline{A} \subset \overline{B}$. □

## 3. Bases and Subbases

Sometimes we don't need the entire collection of open sets to work with a topology. Bases and subbases are smaller subcollections that can suffice.

Definition 2.8. Let $(X, \tau)$ be a topological space and $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a *base* of $\tau$ if $\tau = \{\bigcup_{B \in C} B \mid C \subset \mathcal{B}\}$.

Theorem 2.9. *Let $(X, \tau)$ be a topological space. A collection of sets $\mathcal{B}$ is a base of $\tau$ if and only if the following two conditions are satisfied:*

*(1) $X = \bigcup \mathcal{B}$.*
*(2) For any $B_1, B_2 \in \mathcal{B}$, if $p \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $p \in B_3 \subset B_1 \cap B_2$.*

Proof. Let $\mathcal{B}$ be a base of $\tau$. Therefore $\tau = \{\bigcup_{B \in C} B \mid C \subset \mathcal{B}\}$.

(1) Since $X \in \tau$, there exists a $C \subset \mathcal{B}$ such that $X = \bigcup_{B \in C} B$. Therefore $X \subset \bigcup \mathcal{B}$. Also, since $\bigcup \mathcal{B} \in \tau$, we have $\bigcup \mathcal{B} \subset X$. Therefore $X = \bigcup \mathcal{B}$.
(2) If $B_1, B_2 \in \mathcal{B}$, then $B_1, B_2 \in \tau$. Therefore $B_1 \cap B_2 \in \tau$. Therefore, there exists a $C \subset \mathcal{B}$ such that $B_1 \cap B_2 = \bigcup_{B \in C} B$. This means that if $p \in B_1 \cap B_2$, then there exists a $B_3 \in C$ such that $p \in B_3$. Also $B_3 \subset \bigcup_{B \in C} B$. Therefore we have $p \in B_3 \subset B_1 \cap B_2$.

Conversely, let $\mathcal{B}$ be a collection of sets that satisfies the two conditions. Let $\tau = \{\bigcup_{B \in C} B \mid C \subset \mathcal{B}\}$. We will prove that $\tau$ is a topology on $X$ i.e. it has each of the four properties of Definition 2.1.

(1) Let $A \subset \tau$. Each set in $A$ is a union of a collection of sets in $\mathcal{B}$. Therefore $\bigcup A$ is a union of a collection of sets in $\mathcal{B}$. Therefore $\bigcup A \in \tau$.
(2) Let $A_1, A_2 \in \tau$. There are $C_1, C_2 \subset \mathcal{B}$ such that $A_1 = \bigcup_{X \in C_1} X$ and $A_2 = \bigcup_{Y \in C_2} Y$. Therefore

$$A_1 \cap A_2 = \left(\bigcup_{X \in C_1} X\right) \cap \left(\bigcup_{Y \in C_2} Y\right) = \bigcup_{Y \in C_2} \left(\left(\bigcup_{X \in C_1} X\right) \cap Y\right)$$
$$= \bigcup_{Y \in C_2} \bigcup_{X \in C_1} (X \cap Y)$$

Since $X, Y \in \mathcal{B}$, there exists, for every $p \in X \cap Y$, a $B_p \in \mathcal{B}$ such that $p \in B_p \subset X \cap Y$ (from the second condition). Therefore $X \cap Y = \bigcup_{p \in X \cap Y} B_p$. Therefore $A_1 \cap A_2$ is a union of a collection of sets in $\mathcal{B}$, which proves that $A_1 \cap A_2 \in \tau$.
(3) Since $\phi \subset \mathcal{B}$ and $\bigcup_{B \in \phi} B = \phi$, we have $\phi \in \tau$.
(4) From the first condition, $X \in \tau$.

□

Definition 2.10. Let $(X, \tau)$ be a topological space and $\mathcal{S} \subset \tau$. Then $\mathcal{S}$ is a *subbase* of $\tau$ if the set $\{\bigcap S \mid S \subset \mathcal{S}$ and S is finite$\}$ is a base of $\tau$.

## 4. Density

Definition 2.11. Let $(X, \tau)$ be a topological space and $D \subset X$. Then $D$ is *dense* in $X$ if, for every nonempty open subset $U$ of $X$, $U \cap D$ in nonempty.

Theorem 2.12. *If $(X, \tau)$ is a topological space and $U_1, U_2 \in \tau$ are dense in $X$, then $U_1 \cap U_2$ is dense in $X$.*

Proof. Let $U$ be a nonempty open subset of $X$. Since $U_1$ is dense in $X$, $U \cap U_1 \neq \phi$. Since $U \cap U_1$ is open and $U_2$ is dense in $X$, $U \cap U_1 \cap U_2 \neq \phi$. Since $U$ is arbitrary, this proves that $U_1 \cap U_2$ is dense in $X$. □

## 5. Subspaces

Theorem 2.13. *Let $(X, \tau)$ be a topological space and $Y \subset X$. Let $\tau'$ be a collection of sets with the following property: $V \in \tau'$ if and only if $V = Y \cap U$ for some $U \in \tau$. Then $\tau'$ is a topology on $Y$.*

Proof. We will prove that $\tau'$ has each of the four properties of Definition 2.1.

(1) Let $A \subset \tau'$. For each $V \in A$, $V = Y \cap U$ for some $U \in \tau$. Therefore $\bigcup A = Y \cap (\bigcup B)$ for some $B \subset \tau$. Since $\bigcup B \in \tau$, $\bigcup A \in \tau'$.
(2) Let $A_1, A_2 \in \tau'$. Therefore, there exist $B_1, B_2 \in \tau$ such that $A_1 = Y \cap B_1$ and $A_2 = Y \cap B_2$. Therefore $A_1 \cap A_2 = (Y \cap B_1) \cap (Y \cap B_2) = Y \cap (B_1 \cap B_2)$. Since $B_1 \cap B_2 \in \tau$, $A_1 \cap A_2 \in \tau'$.
(3) Since $\phi = Y \cap \phi$ and $\phi \in \tau$, $\phi \in \tau'$.
(4) Since $Y = Y \cap X$ and $X \in \tau$, $Y \in \tau'$.

□

Definition 2.14. Let $(X, \tau)$ be a topological space and $Y \subset X$. Then $(Y, \tau')$ is a *subspace* of $(X, \tau)$ if $\tau'$ is a topology derived using Theorem 2.13.

Theorem 2.15. *Let $(X, \tau)$ be a topological space and $D \subset Y \subset X$. If $D$ is dense in $X$, then it is also dense in $Y$.*

Proof. Let $V$ be a nonempty open subset of $Y$. Then $V = Y \cap U$ for some nonempty open subset $U$ of $X$. Therefore $D \cap V = D \cap Y \cap U = D \cap U$. Since $D$ is dense in $X$, $D \cap U$ is nonempty. Therefore $D \cap V$ is nonempty. □

## 6. Neighbourhoods

Definition 2.16. Let $(X, \tau)$ be a topological space, $x \in X$ and $N \subset X$. Then $N$ is a *neighbourhood* of $x$ if there exists a set $U \in \tau$ such that $x \in U \subset N$.

The following theorem provides a characterization of open sets in terms of neighbourhoods.

THEOREM 2.17. *Let $(X, \tau)$ be a topological space and $U$ be a subset of $X$. Then $U$ is open if and only if, for every $x \in U$, there is a neighbourhood $N_x$ of $x$ such that $N_x \subset U$.*

PROOF. If $U$ is open, then it is a neighbourhood of every $x \in U$.

Conversely, suppose, for every $x \in U$, there is a neighbourhood $N_x$ of $x$ such that $N_x \subset U$. Then, clearly, for every $x \in U$, there is a $U_x \in \tau$ such that $x \in U_x \subset U$. Moreover $U = \bigcup_x U_x$, which proves that $U \in \tau$. □

## 7. Continuity

DEFINITION 2.18. Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two topolological spaces. A function $f : X \to Y$ is *continuous* at $x \in X$ if, for every neighbourhood $V$ of $f(x)$, there is a neighbourhood $U$ of $x$ such that $f(U) \subset V$. The function $f$ is continuous on the set $X$ if it is continuous at every $x \in X$.

Theorem 2.19 provides a much more practical definition of continuity of a function on a set.

THEOREM 2.19. *Given two topological spaces $(X, \tau_1)$ and $(Y, \tau_2)$, and a function $f : X \to Y$, the following are equivalent.*

*(1) $f$ is continuous on $X$.*
*(2) For every open subset $V$ of $Y$, $f^{-1}(V)$ is an open subset of $X$.*
*(3) For every closed subset $F$ of $Y$, $f^{-1}(F)$ is a closed subset of $X$.*
*(4) For every subset $A$ of $X$, $f(\overline{A}) \subset \overline{f(A)}$.*

PROOF. (1) $\implies$ (2) Let $f$ be continuous on $X$. For any open subset $V$ of $Y$, if $x \in f^{-1}(V)$, then there is an open neighbourhood $U$ of $x$ such that $f(U) \subset V$. However, this implies that $U \subset f^{-1}(V)$, which proves that $f^{-1}(V)$ is open.

(2) $\implies$ (3) If $F$ is a closed subset of $Y$, then $Y - F$ is an open subset of $Y$. Therefore $f^{-1}(Y - F)$ is an open subset of $X$. Therefore $f^{-1}(F) = X - f^{-1}(Y - F)$ is a closed subset of $X$.

(3) $\implies$ (4) For any subset $A$ of $X$, since $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed. Since $A \subset f^{-1}(\overline{f(A)})$, we have $\overline{A} \subset f^{-1}(\overline{f(A)})$. Therefore $f(\overline{A}) \subset \overline{f(A)}$.

(4) $\implies$ (1) For any $x \in X$, let $V$ be an open neighbourhood of $f(x)$. Let $A = X - f^{-1}(V)$ and $U = X - \overline{A}$. We will prove that $U$ is a neighbourhood of $x$ and $f(U) \subset V$.

Since $f(A) \subset Y - V$ and $Y - V$ is closed, $\overline{f(A)} \subset Y - V$. Since $f(x) \notin Y - V$, we have $f(x) \notin \overline{f(A)}$. This implies that $f(x) \notin f(\overline{A})$, since $f(\overline{A}) \subset \overline{f(A)}$. Therefore $x \notin \overline{A}$, which proves that $x \in U$. Since $U$ is open, it is a neighbourhood of x.

Now, for any $y \in U$, we have $y \notin \overline{A}$, which implies that $y \notin A$. Therefore $y \in f^{-1}(V)$ (or $f(y) \in V$). Therefore $f(U) \subset V$. □

## 8. Nets

DEFINITION 2.20. A set $\Lambda$ is *directed* if it has a relation $\leq$ on it such that,

(1) for any $\lambda \in \Lambda$, $\lambda \leq \lambda$,
(2) for any $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$,
(3) for any $\lambda_1, \lambda_2 \in \Lambda$, there exists a $\lambda_3 \in \Lambda$ such that $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

EXAMPLE 2.21. $\mathbb{N}$ is a directed set.

EXAMPLE 2.22. The set $\mathcal{N}_x$ of neighbourhoods of an element $x$ of a topological space is a directed set with the relation $\leq$ defined as follows: If $N_1, N_2 \in \mathcal{N}_x$, then $N_1 \leq N_2$ if $N_2 \subset N_1$.

DEFINITION 2.23. A function $f : \Lambda \to X$ is a *net* in $X$ if $\Lambda$ is directed. For any $\lambda \in \Lambda$, $f(\lambda)$ is denoted as $x_\lambda$. The net itself is denoted as $(x_\lambda)$.

DEFINITION 2.24. A net $(x_\lambda)$ is *eventually* in a set $U$ if there exists a $\lambda_\circ$ such that, for every $\lambda \geq \lambda_\circ$, $x_\lambda \in U$.

DEFINITION 2.25. Let $(X, \tau)$ be a topological space and $x \in X$. A net $(x_\lambda)$ *converges* to $x$ if it is eventually in every neighbourhood of $x$. The convergence of $(x_\lambda)$ to $x$ is denoted as $x_\lambda \to x$.

DEFINITION 2.26. Let $f : \Lambda \to X$ be a net. Let $\phi$ be a directed set such that the net $g : \phi \to \Lambda$ has the following properties:

(1) For every $\mu_1, \mu_2 \in \phi$, if $\mu_1 \leq \mu_2$, then $g(\mu_1) \leq g(\mu_2)$.
(2) For every $\lambda \in \Lambda$, there is a $\mu \in \phi$ such that $\lambda \leq g(\mu)$.

Then $f \circ g$ is a *subnet* of $f$. For any $\mu \in \phi$, $f \circ g(\mu)$ is denotes as $x_{\lambda_\mu}$. The subnet itself is denoted as $(x_{\lambda_\mu})$.

THEOREM 2.27. *If a net $(x_\lambda)$ converges to $x$, then its every subnet also converges to $x$.*

PROOF. Let $(x_{\lambda_\mu})$ be a subnet of $(x_\lambda)$. Let $N$ be a neighbourhood of $x$. If $(x_\lambda)$ converges to $x$, then there exists a $\lambda_\circ$ such that, for every $\lambda \geq \lambda_\circ$, $x_\lambda \in N$.

Now, there exists a $\mu_\circ$ such that $\lambda_{\mu_\circ} \geq \lambda_\circ$. Moreover, for every $\mu \geq \mu_\circ$, $\lambda_\mu \geq \lambda_{\mu_\circ}$. Therefore, for every $\mu \geq \mu_\circ$, $x_{\lambda_\mu} \in N$. Since $N$ is arbitrary, $(x_{\lambda_\mu})$ is eventually in every neighbourhood of $x$. $\square$

The following theorem characterizes continuity in terms of convergence of nets.

THEOREM 2.28. *A function $f : X \to Y$ is continuous at $x \in X$ if and only if, for every net $(x_\lambda)$ in $X$ such that $x_\lambda \to x$, $f(x_\lambda) \to f(x)$.*

PROOF. Suppose $f$ is continuous at $x$. If $V$ is an open neighbourhood of $f(x)$, then $f^{-1}(V)$ is an open neighbourhood of $x$. If $x_\lambda \to x$, then $(x_\lambda)$ is eventually in $f^{-1}(V)$. Therefore $f(x_\lambda)$ is eventually in $V$, which proves that $f(x_\lambda) \to f(x)$.

Conversely suppose $f(x_\lambda) \to f(x)$ for every net $(x_\lambda)$ in $X$ such that $x_\lambda \to x$. Suppose $f$ is not continuous at $x$. Therefore, there exists a neighbourhood $V$ of $f(x)$ such that, for every neighbourhood $U$ of $x$, there exists an $x_U \in U$ such $f(x_U) \notin V$. This implies that the net $(x_U)$ converges to $x$ (Example 2.22), but $(f(x_U))$ does not converge to $f(x)$. This, however, is is a contradiction. Therefore $f$ is continuous at $x$. □

Theorem 2.29. *Let $(X, \tau)$ be a topological space, $F$ be a closed subset of $X$ and $(x_\lambda)$ be a net in $F$. If $x_\lambda \to x$, then $x \in F$.*

Proof. Let $x \in X - F$. Since $X - F$ is open, there is a neighbourhood $N$ of $x$ such that $N \subset X - F$ (Theorem 2.17). Also, since $x_\lambda \to x$, $(x_\lambda)$ is eventually in $N$. This, however, is contradiction, since $(x_\lambda)$ is in $F$. □

## 9. Separation

In this section, we will define topological spaces whose elements are separated from one another in a certain manner.

Definition 2.30. A topological space $(X, \tau)$ is a *$T_1$-space* if, for every $x, y \in X$ such that $x \neq y$, there exist open subsets $U$ and $V$ of $X$ such that $x \in U$ but $y \notin U$, and $y \in V$ but $x \notin V$.

Definition 2.31. A topological space $(X, \tau)$ is *Hausdorff* if, for every $x, y \in X$ such that $x \neq y$, there exist disjoint open subsets $U$ and $V$ of $X$ such that $x \in U$ and $y \in V$.

It can be observed that every Hausdorff space is also a $T_1$-space.

Example 2.32. Every metrizable space (Definition 3.6) is Hausdorff.

Definition 2.33. A $T_1$-space $(X, \tau)$ is *normal* if, for any disjoint closed subsets $F_1$ and $F_2$ of $X$, there exist disjoint open subsets $U_1$ and $U_2$ of $X$ such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

## 10. Compactness

Definition 2.34. Let $(X, \tau)$ be a topological space and $A \subset X$. A collection of sets $\mathcal{U}$ is an *open cover* of $A$ if $\mathcal{U} \subset \tau$ and $A \subset \bigcup \mathcal{U}$.

Definition 2.35. A set $A$ is *compact* if every open cover of $A$ contains a finite subcover of $A$.

Theorem 2.36. *Let $(X, \tau)$ be a topological space and $A \subset X$. If $X$ is compact and $A$ is closed, then $A$ is compact.*

Proof. Let $\mathcal{U}$ be any open cover of $A$. If $A$ is closed, then $X - A$ is open. Therefore $\mathcal{U} \cup \{X - A\}$ is an open cover of $X$. Since $X$ is compact, there exist $U_1, \ldots, U_n \in \mathcal{U} \cup \{X - A\}$ such that $X \subset U_1 \cup \cdots \cup U_n$. If $X - A \notin \{U_1, \ldots, U_n\}$, then $\{U_1, \ldots, U_n\}$ is a finite subcover of $A$ in $\mathcal{U}$. Otherwire, $\{U_1, \ldots, U_n\} - \{X - A\}$ is a finite subcover of $A$ in $\mathcal{U}$. Therefore $A$ is compact. □

THEOREM 2.37. *Let $(X,\tau)$ be a Hausdorff space and $A \subset X$. If $A$ is compact, then it is closed.*

PROOF. Consider an arbitrary $x \in X - A$. Since $(X,\tau)$ is Hausdorff, for any $y \in A$, there are $U_y, V_y \in \tau$ such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$.

Now $V = \{V_y : y \in A\}$ is an open cover of $A$. Since $A$ is compact, there are $V_{y_1}, \ldots, V_{y_n} \in V$ such that $A \subset V_{y_1} \cup \cdots \cup V_{y_n}$. Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$. Then $U$ is a neighbourhood of $x$. Also, $(U \cap A) \subset U \cap (V_{y_1} \cup \cdots \cup V_{y_n}) = \phi$. Therefore $U \subset X - A$. Since $x$ is arbitrary, $X - A$ is open (Theorem 2.17), which proves that $A$ is closed. □

The following theorem shows that a continuous function maps a compact set to a compact set.

THEOREM 2.38. *Let $(X,\tau)$ and $(Y,\tau)$ be topological spaces, $f : X \to Y$ be a continuous function and $A \subset X$. If $A$ is compact, then $f(A)$ is also compact.*

PROOF. Let $\mathcal{U}$ be any open cover of $f(A)$. Since $f$ is continuous, $f^{-1}(U)$ is open for every $U \in \mathcal{U}$. Therefore, $U_A = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of $A$. Since A is compact, there are $f^{-1}(U_1), \ldots, f^{-1}(U_n) \in U_A$ such that $A \subset f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)$. However, this implies that $f(A) \subset U_1 \cup \cdots \cup U_n$, which proves that $f(A)$ is compact. □

## 11. Urysohn's Lemma

Urysohn's Lemma shows the existence of certain continuous functions on a normal topological space.

LEMMA 2.39. *Let $(X,\tau)$ be a normal topological space, $U$ be an open subset of $X$, $F$ be a closed subset of $X$ and $F \subset U$. Then there exists an open subset $V$ of $X$ such that*

$$F \subset V \subset \overline{V} \subset U$$

PROOF. Since $U$ is open and $F \subset U$, $X - U$ is closed, and $F$ and $X - U$ are disjoint. Since $(X,\tau)$ is normal, there exist disjoint open sets $V$ and $W$ such that $F \subset V$ and $X - U \subset W$. Since $V$ and $W$ are disjoint, $V \subset X - W$. Since $X - W$ is closed, $\overline{V} \subset X - W$. However, since $X - U \subset W$, $X - W \subset U$. Therefore $\overline{V} \subset U$. This proves that $F \subset V \subset \overline{V} \subset U$. □

THEOREM 2.40 (Urysohn's Lemma). *Let $(X,\tau)$ be a normal topological space. For every pair of disjoint closed subsets, $F$ and $G$, of $X$, there exists a continuous function $f : X \to [0,1]$ such that $f(F) = 0$ and $f(G) = 1$.*

PROOF. Let $D$ be the set of all rational numbers of the form $\frac{m}{2^n}$, where $m, n \in \mathbb{N}$ and $m < 2^n$. We construct a function $g : D \to \tau$ in the following manner.

Since $G$ is closed and $F$ and $G$ are disjoint, $X - G$ is open and $F \subset X - G$. Therefore, from Lemma 2.39, there exists an open set, say $U_{\frac{1}{2}}$, such that

$$F \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset X - G$$

Let $g(\frac{1}{2}) = U_{\frac{1}{2}}$. Similarly, there exist open sets, say $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$, such that

$$F \subset U_{\frac{1}{4}} \subset \overline{U_{\frac{1}{4}}} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_{\frac{3}{4}} \subset \overline{U_{\frac{3}{4}}} \subset X - G$$

Let $g(\frac{1}{4}) = U_{\frac{1}{4}}$ and $g(\frac{3}{4}) = U_{\frac{3}{4}}$. And so on.

The function $g$ constructed above has the following property. For any $d_1, d_2 \in D$ such that $d_1 < d_2$,

$$F \subset g(d_1) \subset \overline{g(d_1)} \subset g(d_2) \subset \overline{g(d_2)} \subset X - G$$

Now let a function $f : X \to [0, 1]$ be defined as follows. For any $x \in X$,

$$f(x) = \begin{cases} sup(d \in D; x \notin g(d)) & x \notin \bigcap_{d \in D} g(d) \\ 0 & x \in \bigcap_{d \in D} g(d) \end{cases}$$

Here $f(F) = 0$ and $f(G) = 1$. We will prove that $f$ is continuous.

Consider an arbitrary $a \in (0, 1)$. For any $x \in X$, if $f(x) < a$, then there exists a $d \in D$ such that $d < a$ and $x \in g(d)$. Conversly, for any $d \in D$ such that $d < a$, if $x \in g(d)$, then $f(x) < a$. Therefore $f^{-1}([0, a)) = \bigcup_{d<a} g(d)$, which implies that $f^{-1}([0, a))$ is open.

For any $x \in X$, if $f(x) > a$, then there exists a $d \in D$ such that $d > a$ and $x \notin \overline{g(d)}$. Conversly, for any $d \in D$ such that $d > a$, if $x \notin \overline{g(d)}$, then $f(x) > a$. Therefore $f^{-1}((a, 1]) = \bigcup_{d>a}(X - \overline{g(d)})$, which is open.

Since $a$ is arbitrary and the set of intervals of the form $[0, a)$ or $(a, 1]$ is a subbase of $[0, 1]$, $f^{-1}(V)$ is open for any open subset $V$ of $[0, 1]$. Therefore $f$ is continuous (Theorem 2.19). □

It can be observed that, in Urysohn's Lemma, the interval $[0, 1]$ can be replaced with any closed interval of real numbers.

CHAPTER 3

# Topics in Metric Spaces

In this chapter, we will look at a few concepts in metric space theory, which will be needed to discuss the application of Topology to metric spaces in the next chapter.

## 1. Metric Spaces

We begin with the definition of a metric space.

DEFINITION 3.1. Given a set $X$, a function $d : X \times X \to \mathbb{R}$ is a *metric* on $X$ if, for every $x, y, z \in X$,

(1) $d(x, y) \geq 0$,
(2) $d(x, y) = 0$ if and only if $x = y$,
(3) $d(x, y) = d(y, x)$,
(4) $d(x, y) \leq d(x, z) + d(z, y)$.

EXAMPLE 3.2. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as $d(x, y) = |x - y|$ for every $x, y \in \mathbb{R}$, is a metric on $\mathbb{R}$. This is the usual metric on $\mathbb{R}$.

DEFINITION 3.3. A tuple $(X, d)$ is a *metric space* if $d$ is a metric on $X$.

## 2. Metrizable Spaces

The theories of topology and metric spaces can be linked together with the concept of open balls.

DEFINITION 3.4. Let $(X, d)$ be a metric space, $x \in X$ and $\epsilon \in \mathbb{R}$. Then the set of all $y \in X$ such that $d(x, y) < \epsilon$ is an *open ball* of radius $\epsilon$ centered at $x$. It is denoted as $B_d(x, \epsilon)$.

THEOREM 3.5. *Let $\mathcal{B}$ be the set of open balls in a metric space $(X, d)$. Then $\mathcal{B}$ is a base of a topology on $X$.*

PROOF. If $\mathcal{B}$ is the set of open balls in $(X, d)$, then $X = \bigcup \mathcal{B}$.

Now let $B_d(x_1, \epsilon_1), B_d(x_2, \epsilon_2) \in \mathcal{B}$. For any $x \in B_d(x_1, \epsilon_1) \cap B_d(x_2, \epsilon_2)$, let $\epsilon = min(\epsilon_1 - d(x, x_1), \epsilon_2 - d(x, x_2))$. For any $y \in B_d(x, \epsilon)$,

$$\begin{aligned} d(y, x_1) &\leq d(y, x) + d(x, x_1) \\ &< \epsilon + d(x, x_1) \\ &\leq \epsilon_1 - d(x, x_1) + d(x, x_1) \\ &= \epsilon_1 \end{aligned}$$

Therefore $y \in B_d(x_1, \epsilon_1)$. Similarly it can be proven that $y \in B_d(x_2, \epsilon_2)$. Since $x$ and $y$ are arbitrary, $B_d(x, \epsilon) = B_d(x_1, \epsilon_1) \cap B_d(x_2, \epsilon_2)$. Therefore $\mathcal{B}$ is a base of a topology on $X$ (Theorem 2.9). □

DEFINITION 3.6. A topological space $(X, \tau)$ is *metrizable* if $\tau$ has been derived from a metric space $(X, d)$ using Theorem 3.5.

From this point on, whenever we talk about a topological property of a metric space, such as convergence, continuity, density etc., we would mean the topological property of the corresponding metrizable space.

THEOREM 3.7. *If $U$ is an open set in a metric space $(X, d)$, then, for every $x \in U$, there is an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$.*

PROOF. If $U$ is open, then $x \in U$ implies that $x \in B_d(x_0, \delta) \subset U$ for some $x_0 \in U$ and $\delta > 0$. Let $\epsilon = \delta - d(x, x_0)$. Therefore, if $y \in B_d(x, \epsilon)$, then

$$\begin{aligned} d(y, x_0) &\leq d(y, x) + d(x, x_0) \\ &< \epsilon + d(x, x_0) \\ &= \delta - d(x, x_0) + d(x, x_0) \\ &= \delta \end{aligned}$$

Therefore $B_d(x, \epsilon) \subset B_d(x_0, \delta) \subset U$. □

## 3. Continuity in Metric Spaces

Continuity, a topological property, was discussed in the previous chapter. Here, we provide a few additional results as they relate to metric spaces.

THEOREM 3.8. *Given metric spaces $(X_1, d_1)$ and $(X_2, d_2)$, a function $f : X_1 \to X_2$, and $x \in X_1$, the following are equivalent:*

*(1) $f$ is continuous at $x$.*
*(2) For every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(B_{d_1}(x, \delta)) \subset B_{d_2}(f(x), \epsilon)$.*

PROOF. (1) $\Longrightarrow$ (2) For any $\epsilon > 0$, $B_{d_2}(f(x), \epsilon)$ is open. Therefore it is a neighbourhood of $f(x)$. Since $f$ is continuous, there exists an open neighbourhood $U$ of $x$ in $X_1$ such that $f(U) \subset B_{d_2}(f(x), \epsilon)$. Since $U$ is open, there is a $\delta > 0$ such that $B_{d_1}(x, \delta) \subset U$ and, therefore, $f(B_{d_1}(x, \delta)) \subset B_{d_2}(f(x), \epsilon)$.

(2) $\Longrightarrow$ (1) Let $V$ be an open neighbourhood of $f(x)$. Therefore, there exists an $\epsilon > 0$ such that $B_{d_2}(f(x), \epsilon) \subset V$. There also exists a $\delta > 0$ such that $f(B_{d_1}(x, \delta)) \subset B_{d_2}(f(x), \epsilon)$ and, therefore, $f(B_{d_1}(x, \delta)) \subset V$. Since $B_{d_1}(x, \delta)$ is open, it is a neighbourhood of $x$, which proves that $f$ is continuous. □

DEFINITION 3.9. Let $d$ and $\tilde{d}$ be metrics on a set $X$. Then $\tilde{d}$ is *equivalent* to $d$ if the identity map on $X$ is continuous both from $(X, d)$ to $(X, \tilde{d})$ and from $(X, \tilde{d})$ to $(X, d)$.

THEOREM 3.10. *Let $d$ and $\tilde{d}$ be metrics on a set $X$, and $\tau$ and $\tilde{\tau}$ be topologies derived from $(X, d)$ and $(X, \tilde{d})$ respectively using Theorem 3.5. If $\tilde{d}$ is equivalent to $d$, then $\tilde{\tau} = \tau$.*

PROOF. Let $U \in \tau$. Since $\tilde{d}$ is equivalent to $d$, the identity map on $X$ is continuous from $(X, \tilde{d})$ to $(X, d)$. Therefore $U \in \tilde{\tau}$ (Theorem 2.19). This implies that $\tau \subset \tilde{\tau}$.

It can be proven in an identical manner that $\tilde{\tau} \subset \tau$. □

## 4. Sequences and Completeness

DEFINITION 3.11. A net is a *sequence* if its domain is $\mathbb{N}$.

THEOREM 3.12. *If $(x_n)$ is a sequence in a metric space $(X, d)$ and $x \in X$, then the following are equivalent:*

*(1) $(x_n)$ converges to $x$.*
*(2) For every $\epsilon > 0$, $(x_n)$ is eventually in $B_d(x, \epsilon)$.*

PROOF. (1) $\implies$ (2) For every $\epsilon > 0$, $B_d(x, \epsilon)$ is open. Therefore it is a neighbourhood of $x$. Since $x_n$ converges to $x$, $(x_n)$ is eventually in $B_d(x, \epsilon)$.

(2) $\implies$ (1) Let $N$ be a neighbourhood of $x$. Therefore, there is an open set $U$ such that $x \in U \subset N$. Since $U$ is open, there is an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$ (Theorem 3.7). Since $(x_n)$ is eventually in $B_d(x, \epsilon)$, it is eventually in $N$. □

THEOREM 3.13. *Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces and $x \in X$. Then a function $f : X \to Y$ is continuous at $x$ if and only if the following statement is true: For every sequence $(x_n)$ in $X$ such that $x_n \to x$, $f(x_n) \to f(x)$.*

PROOF. If $f$ is continuous at $x$, then the statement is true (Theorem 2.28).

Conversely, let the statement be true. Suppose $f$ is not continuous at $x$. Therefore, there exists an $\epsilon > 0$, such that, for every $\delta > 0$, $f(B_{d_1}(x, \delta)) \not\subset B_{d_2}(f(x), \epsilon)$ (Theorem 3.8). Therefore, for every $n \in \mathbb{N}$, there exists an $x_n \in B_{d_1}(x, \frac{1}{n})$ such that $f(x_n) \notin B_{d_2}(f(x), \epsilon)$. For every $\delta > 0$, the sequence $(x_n)$ is eventually in $B_{d_1}(x, \delta)$. However, $f(x_n)$ is never in $B_{d_2}(f(x), \epsilon)$. This is a contradiction. □

DEFINITION 3.14. A sequence $(x_n)$ in a metric space $(X, d)$ is *Cauchy* if, for every $\epsilon > 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that, for every $m, n \geq N_\epsilon$, $d(x_m, x_n) < \epsilon$.

DEFINITION 3.15. A metric space $(X, d)$ is *complete* if every Cauchy sequence converges in it.

EXAMPLE 3.16. $\mathbb{R}$ with its usual metric (Example 3.2) is a complete metric space.

CHAPTER 4

# Application of Topology to Metric Spaces

In this chapter, we will apply the topological concepts discussed in Chapter 2 to metric spaces to derive some interesting results, namely Cantor's Intersection Theorem, Bourbaki's Mittag-Leffler Theorem and Baire's Theorem.

We begin with a few essential definitions.

Definition 4.1. A metric space $(X, d)$ is *bounded* if there exists an $r \in \mathbb{R}$ such that, for every $x, y \in X$, $d(x, y) \leq r$.

Definition 4.2. If $(X, d)$ is a bounded metric space, then $\sup\limits_{x,y \in X} d(x, y)$ is its *diameter*. It is denoted as $diam(X)$.

Theorem 4.3 (Cantor's Intersection Theorem). *Let $(X, d)$ be a complete metric space and $(F_n)$ be a sequence of nonempty closed subsets of $X$ such that $diam(F_n) \to 0$ and, for every $n \in \mathbb{N}$, $F_{n+1} \subset F_n$. Then there is one and only one element in $\bigcap F_n$.*

Proof. For every $n \in \mathbb{N}$, let $x_n \in F_n$. Let $\epsilon$ be a positive real number. Since $diam(F_n) \to 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that, for every $n \geq N_\epsilon$, $diam(F_n) \in B_d(0, \epsilon)$ (Theorem 3.12). Therefore $diam(F_{N_\epsilon}) < \epsilon$. Now we know that, for every $n \in \mathbb{N}$, $F_{n+1} \subset F_n$. Therefore, for every $m, n \geq N_\epsilon$, $x_m, x_n \in F_{N_\epsilon}$. This implies that $d(x_m, x_n) < \epsilon$. Since $\epsilon$ is arbitrary, this proves that $(x_n)$ is Cauchy. Since $(X, d)$ is complete, there exists an $x \in X$ such that $x_n \to x$.

For any $n \in \mathbb{N}$, let $(x_{n_k})$ be the subsequence of $(x_n)$ such that $n_k = n + k$. Since $x_n \to x$, $x_{n_k} \to x$ (Theorem 2.27). Moreover, $(x_{n_k}) \subset F_n$. Since $F_n$ is closed, $x \in F_n$ (Theorem 2.29). Since $n$ is arbitrary, $x \in \bigcap F_n$.

Let $x' \in \bigcap F_n$ such that $x' \neq x$. Therefore $d(x', x) > 0$. Since $diam(F_n) \to 0$, there exists an $n \in \mathbb{N}$ such that $diam(F_n) < d(x', x)$ (Theorem 3.12). However, since $x', x \in F_n$, $d(x', x) \leq diam(F_n)$. This is a contradiction. Therefore $x$ is the only element in $\bigcap F_n$. □

The following lemma is needed to prove Bourbaki's Mittag-Leffler Theorem.

Lemma 4.4. *Let $(X_1, d_1)$ and $(X_2, d_2)$ be metric spaces, $(X_1, d_1)$ be complete, and $f : X_1 \to X_2$ be a continuous function. Then a metric $\tilde{d}_1$ can be defined on $X_1$ as follows: For any $x, y \in X_1$, $\tilde{d}_1(x, y) = d_1(x, y) + d_2(f(x), f(y))$. Moreover,*

*(1) $\tilde{d}_1$ is equivalent to $d_1$, and*
*(2) $(X_1, \tilde{d}_1)$ is complete.*

PROOF. First, we will prove that $\tilde{d}_1$ is a metric on $X_1$ i.e. it has each of the four properties of Definition 3.1. Let $x, y, z \in X_1$.

(1) It is obvious that $\tilde{d}_1(x, y) \geq 0$.
(2) If $\tilde{d}_1(x, y) = 0$, then $d_1(x, y) = 0$, which implies that $x = y$. Conversely, if $x = y$, then $\tilde{d}_1(x, y) = 0$.
(3) It is obvious that $\tilde{d}_1(x, y) = \tilde{d}_1(y, x)$.
(4) The fourth property can be shown as follows:

$$\begin{aligned}\tilde{d}_1(x, y) &= d_1(x, y) + d_2(f(x), f(y)) \\ &\leq d_1(x, z) + d_1(z, y) + d_2(f(x), f(z)) + d_2(f(z), f(y)) \\ &= (d_1(x, z) + d_2(f(x), f(z))) + (d_1(z, y) + d_2(f(z), f(y))) \\ &= \tilde{d}_1(x, z) + \tilde{d}_1(z, y)\end{aligned}$$

Now we will prove that $\tilde{d}_1$ is equivalent to $d_1$. For any $x, y \in X_1$, $d_1(x, y) \leq \tilde{d}_1(x, y)$. Therefore, for any $x \in X_1$ and $\epsilon > 0$, $B_{\tilde{d}_1}(x, \epsilon) \subset B_{d_1}(x, \epsilon)$. Therefore, the identity map on $X_1$ is continuous from $(X_1, \tilde{d}_1)$ to $(X_1, d_1)$ (Theorem 3.8).

Let $\epsilon$ be an arbitrary positive real number. Since $f$ is continuous, there exists a $\delta_x$ for every $x \in X_1$ such that $f(B_{d_1}(x, \delta_x)) \subset B_{d_2}(f(x), \frac{\epsilon}{2})$ (Theorem 3.8). Let $\delta = min(\delta_x, \frac{\epsilon}{2})$. If $y \in B_{d_1}(x, \delta)$, then $y \in B_{d_1}(x, \delta_x)$, which implies that $f(y) \in B_{d_2}(f(x), \frac{\epsilon}{2})$. Therefore

$$\begin{aligned}\tilde{d}_1(x, y) &= d_1(x, y) + d_2(f(x), f(y)) \\ &< \delta + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

Therefore $B_{d_1}(x, \delta) \subset B_{\tilde{d}_1}(x, \epsilon)$. Therefore the identity map on $X_1$ is continuous from $(X_1, d_1)$ to $(X_1, \tilde{d}_1)$ (Theorem 3.8).

Finally, we will prove that $(X_1, \tilde{d}_1)$ is complete. Let $(x_n)$ be an arbitrary Cauchy sequence in $(X_1, \tilde{d}_1)$. Since, for every $x, y \in X_1$, $d_1(x, y) \leq \tilde{d}_1(x, y)$, $(x_n)$ is Cauchy in $(X_1, d_1)$. Since $(X_1, d_1)$ is complete, $(x_n)$ is convergent in it. Since $\tilde{d}_1$ is equivalent to $d_1$, $(x_n)$ is convergent in $(X_1, \tilde{d}_1)$ (Theorem 3.10), which proves the completeness of $(X_1, \tilde{d}_1)$. □

THEOREM 4.5 (Bourbaki's Mittag-Leffler Theorem). *Let $(X_n, d_n)$ be a sequence of metric spaces and $(f_n : X_{n+1} \to X_n)$ a sequence of functions with the following properties for every $n \in \mathbb{N}$:*

*(1) $(X_n, d_n)$ is complete.*
*(2) $f_n$ is continuous.*
*(3) $f_n(X_{n+1})$ is dense in $X_n$.*

*Then,*

$$D = \bigcap_{n\in\mathbb{N}} (f_1 \circ \cdots \circ f_n)(X_{n+1})$$

*is dense in* $X_1$.

Proof. We begin by using Lemma 4.4 to inductively replace, for every $n \in \mathbb{N}$, the metric $d_n$ with the metric $\tilde{d}_n$ defined as follows: $\tilde{d}_1$ is just $d_1$ and, for any $x, y \in X_{n+1}$,

$$\tilde{d}_{n+1}(x,y) = d_{n+1}(x,y) + \tilde{d}_n(f_n(x), f_n(y))$$

The lemma assures that, for every $n \in \mathbb{N}$, $(X_n, \tilde{d}_n)$ is still complete, and, since $\tilde{d}_n$ is equivalent to $d_n$, $f_n$ is still continuous and $f_n(X_{n+1})$ still dense in $X_n$. But most importantly, for every $x, y \in X_{n+1}$,

$$\tilde{d}_n(f_n(x), f_n(y)) \leq \tilde{d}_{n+1}(x,y)$$

This property is the reason why we replaced the metrics. We will need it later in the proof.

Now, if $U_0$ is an arbitrary non-empty open set in $X_1$, then we have to show that $U_0 \cap D \neq \phi$. This can be done by constructing a sequence of open sets in the following manner.

Since $f_1(X_2)$ is dense in $X_1$, $U_0 \cap f_1(X_2) \neq \phi$. Therefore, there exists an $x_2 \in X_2$ such that $f_1(x_2) \in U_0$. Since $U_0$ is open, it is a neighbourhood of $f_1(x_2)$, and, therefore, there exists an $\epsilon > 0$ such that $B_{\tilde{d}_1}(f_1(x_2), \epsilon) \subset U_0$ (Theorem 3.7). Since $f_1$ is continuous, there exists a $\delta > 0$ such that $f_1(B_{\tilde{d}_2}(x_2, \delta)) \subset B_{\tilde{d}_1}(f_1(x_2), \frac{\epsilon}{2})$ and $\delta \leq 1$ (Theorem 3.8). In this case, $\overline{f_1(B_{\tilde{d}_2}(x_2,\delta))} \subset \overline{B_{\tilde{d}_1}(f_1(x_2), \frac{\epsilon}{2})} \subset B_{\tilde{d}_1}(f_1(x_2), \epsilon) \subset U_0$ (Theorem 2.7). Let $U_1 = B_{\tilde{d}_2}(x_2, \delta)$. Therefore, $diam(U_1) \leq 1$ and $\overline{f_1(U_1)} \subset U_0$.

Since $U_1$ is an open set in $X_2$, we can similarly obtain an open set $U_2$ in $X_3$ such that $diam(U_2) \leq \frac{1}{2}$ and $\overline{f_2(U_2)} \subset U_1$. Thus, we can obtain a sequence $(U_n)$ of open sets such that $diam(U_n) \leq \frac{1}{n}$ and $\overline{f_n(U_n)} \subset U_{n-1}$.

For any $m \geq 0$, let $(Y_n^m)$ be a sequence of sets in $U_m$ such that

$$Y_n^m = \overline{(f_{m+1} \circ \cdots \circ f_{m+n})(U_{m+n})}$$

This sequence has the following properties:

1. For every $n \in \mathbb{N}$, $Y_n^m$ is closed.
2. For every $n \in \mathbb{N}$, $Y_{n+1}^m \subset Y_n^m$ (Theorem 2.7).
3. We know that, for every $n \in \mathbb{N}$ and every $x, y \in X_{n+1}$, $\tilde{d}_n(f_n(x), f_n(y)) \leq \tilde{d}_{n+1}(x,y)$. This implies that, for every $n \in \mathbb{N}$, $diam(Y_n^m) \leq \frac{1}{m+n}$. Therefore $diam(Y_n^m) \to 0$.

Therefore, there exits a unique $y_m \in \bigcap_{n\in\mathbb{N}} Y_n^m$ (Theorem 4.3).

For every $m \geq 0$, $f_{m+1}$ is continuous. Therefore $f_{m+1}(Y_n^{m+1}) \subset Y_{n+1}^m$ (Theorem 2.19). Since, for every $n \in \mathbb{N}$, $y_{m+1} \in Y_n^{m+1}$, $f_{m+1}(y_{m+1}) \in Y_{n+1}^m$. Moreover $y_{m+1} \in U_{m+1}$, which implies that $f_{m+1}(y_{m+1}) \in \overline{f_{m+1}(U_{m+1})}$. Therefore

$$\begin{aligned} f_{m+1}(y_{m+1}) &\in \overline{f_{m+1}(U_{m+1})} \cap \bigcap_{n\in\mathbb{N}} Y^m_{n+1} \\ &= Y^m_1 \cap \bigcap_{n\in\mathbb{N}} Y^m_{n+1} \\ &= \bigcap_{n\in\mathbb{N}} Y^m_n \end{aligned}$$

This shows that, for every $m \geq 0$, $y_m = f_{m+1}(y_{m+1})$. Therefore, there exists a $y_0 \in U_0$ such that, for every $n \in \mathbb{N}$, $y_0 \in (f_1 \circ \cdots \circ f_n)(X_{n+1})$. Therefore $y_0 \in U_0 \cap D$. □

We need another definition and a couple of lemmas before we can prove Baire's Theorem.

Definition 4.6. Let $(X, d)$ be a metric space, $x \in X$ and $A \subset X$. Then, $\inf\limits_{y\in A} d(x, y)$ is the *distance* of $x$ to $A$. It is denoted as $dist(x, A)$.

Lemma 4.7. *Let $(X, d)$ be a metric space, $A \subset X$ and $(x_n)$ a sequence in $X$. If $x_n \to x$, then $dist(x_n, A) \to dist(x, A)$.*

Proof. For every $n \in \mathbb{N}$ and $a \in A$,

$$\begin{aligned} &d(x_n, a) \leq d(x_n, x) + d(x, a) \\ \Longrightarrow\ &\inf_{y\in A} d(x_n, y) \leq d(x_n, x) + d(x, a) \\ \Longrightarrow\ &\inf_{y\in A} d(x_n, y) - d(x_n, x) \leq d(x, a) \\ \Longrightarrow\ &\inf_{y\in A} d(x_n, y) - d(x_n, x) \leq \inf_{y\in A} d(x, y) \\ \Longrightarrow\ &\inf_{y\in A} d(x_n, y) - \inf_{y\in A} d(x, y) \leq d(x_n, x) \\ \Longrightarrow\ &dist(x_n, A) - dist(x, A) \leq d(x_n, x) \end{aligned}$$

Similarly it can be proven that

$$dist(x, A) - dist(x_n, A) \leq d(x_n, x)$$

Therefore

$$|dist(x_n, A) - dist(x, A)| \leq d(x_n, x)$$

which proves that $dist(x_n, A) \to dist(x, A)$. □

Lemma 4.8. *Let $(X, d)$ be a complete metric space, $U$ be an open subset of $X$, and $d_U : U \times U \to \mathbb{R}$ be defined as follows: For $x, y \in U$,*

$$d_U(x, y) = d(x, y) + \left| \frac{1}{dist(x, X - U)} - \frac{1}{dist(y, X - U)} \right|$$

*Then,*

*(1) $d_U$ is a metric on $U$,*

*(2)* $d_U$ *is equivalent to* $d$, *and*
*(3)* $(U, d_U)$ *is complete.*

PROOF. First we will prove that $d_U$ is well defined i.e. $dist(x, X-U) \neq 0$ for every $x \in U$. If $dist(x, X-U) = 0$ for some $x \in U$, then, for every $\epsilon > 0$, there is a $y \in X-U$ such that $d(x,y) < \epsilon$. However, since $U$ is open, there is an $\epsilon$ such that $B_d(x,\epsilon) \subset U$ (Theorem 3.7). This is a contradiction.

Now we will prove that $d_U$ has each of the four properties of Definition 3.1. Let $x, y, z \in U$.

(1) It is obvious that $d_U(x,y) \geq 0$.
(2) If $d_U(x,y) = 0$, then $d(x,y) = 0$, which implies that $x = y$. Conversely, if $x = y$, then $d_U(x,y) = 0$.
(3) It is obvious that $d_U(x,y) = d_U(y,x)$.
(4) The fourth property can be shown as follows:

$$\begin{aligned}
d_U(x,y) &= d(x,y) + \left|\frac{1}{dist(x,X-U)} - \frac{1}{dist(y,X-U)}\right| \\
&\leq d(x,z) + d(z,y) + \left|\frac{1}{dist(x,X-U)} - \frac{1}{dist(z,X-U)}\right| \\
&\quad + \left|\frac{1}{dist(z,X-U)} - \frac{1}{dist(y,X-U)}\right| \\
&= d(x,z) + \left|\frac{1}{dist(x,X-U)} - \frac{1}{dist(z,X-U)}\right| \\
&\quad + d(z,y) + \left|\frac{1}{dist(z,X-U)} - \frac{1}{dist(y,X-U)}\right| \\
&= d_U(x,z) + d_U(z,y)
\end{aligned}$$

Therefore $d_U$ is a metric on $U$.

Now we will prove that $d_U$ is equivalent to $d$. For any $x, y \in U$, $d(x,y) \leq d_U(x,y)$. Therefore, for any $x \in U$ and $\epsilon > 0$, $B_{d_U}(x,\epsilon) \subset B_d(x,\epsilon)$. Therefore, the identity map on $U$ is continuous from $(U, d_U)$ to $(U, d)$ (Theorem 3.8).

For any $x \in U$, let $(x_n)$ be a sequence in $U$ such that $x_n \to x$ in $(U,d)$. For any $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that, for every $n \geq N_1$, $d(x_n, x) < \frac{\epsilon}{2}$ (Theorem 3.12). Also, from Lemma 4.7, $dist(x_n, X-U) \to dist(x, X-U)$, which implies that $\frac{1}{dist(x_n,X-U)} \to \frac{1}{dist(x,X-U)}$. Therefore, there exists an $N_2 \in \mathbb{N}$ such that, for every $n \geq N_2$, $\left|\frac{1}{dist(x_n,X-U)} - \frac{1}{dist(x,X-U)}\right| < \frac{\epsilon}{2}$ (Theorem 3.12). Let $N = max(N_1, N_2)$. For every $n \geq N$,

$$\begin{aligned}
d_U(x_n,x) &= d(x_n,x) + \left|\frac{1}{dist(x_n,X-U)} - \frac{1}{dist(x,X-U)}\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore $x_n \to x$ in $(U, d_U)$ also (Theorem 3.12). This proves that the identity map on $U$ is continuous from $(U, d)$ to $(U, d_U)$ (Theorem 3.13).

Now we will prove that $(U, d_U)$ is complete. Let $(x_n)$ be a Cauchy sequence in $(U, d_U)$. Since $d(x, y) \leq d_U(x, y)$ for every $x, y \in U$, $(x_n)$ is Cauchy in $(X, d)$ and, therefore, convergent. Let $x_n \to x$. First we will prove that $x \in U$.

Suppose $x \in X - U$. This implies that $dist(x_n, X - U) \to 0$ (Lemma 4.7). Since $(x_n)$ is Cauchy in $(U, d_U)$ and, for every $m, n \in \mathbb{N}$, $\left|\frac{1}{dist(x_m, X-U)} - \frac{1}{dist(x_n, X-U)}\right| \leq d_U(x_m, x_n)$, $\left(\frac{1}{dist(x_n, X-U)}\right)$ is a Cauchy sequence in $\mathbb{R}$. Therefore, $\left(\frac{1}{dist(x_n, X-U)}\right)$ is convergent in $\mathbb{R}$. This, however, is is a contradiction since $dist(x_n, X - U) \to 0$. Therefore $x \in U$.

Since $d_U$ is equivalent to $d$, $x_n \to x$ in $(U, d_U)$ also, which proves that $(U, d_U)$ is complete. □

Theorem 4.9 (Baire's Theorem). *Let $(X, d)$ be a complete metric space and $(U_n)$ a sequence of dense open subsets of $X$. Then, $\bigcap_{n=1}^{\infty} U_n$ is also dense in $X$.*

Proof. For any $n \in \mathbb{N}$, let $V_n = U_1 \cap \cdots \cap U_n$. The sequence $(V_n)$ has the following properties:

(1) For every $n \in \mathbb{N}$, $V_{n+1} \subset V_n$.
(2) $\bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n$.

Let $f_1 : V_1 \to X$ be defined in the following manner: For any $x \in V_1$, $f_1(x) = x$. For $n > 1$, let $f_n : V_n \to V_{n-1}$ be defined in a similar manner. Therefore, for every $n \in \mathbb{N}$, $V_n = (f_1 \circ \cdots \circ f_n)(V_n)$, which implies

$$\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n \in \mathbb{N}} (f_1 \circ \cdots \circ f_n)(V_n)$$

For any $n \in \mathbb{N}$, let $d_{V_n}$ be the metric on $V_n$ as defined in Lemma 4.8. Then the sequence of metric spaces $(V_n, d_{V_n})$ and the sequence of functions $(f_n)$ have the following properties:

(1) For every $n \in \mathbb{N}$, $(V_n, d_{V_n})$ is complete (Lemma 4.8).
(2) The function $f_1$ is continuous from $(V_1, d)$ to $(X, d)$. Since $d_{V_1}$ is equivalent to $d$, $f_1$ is also continuous from $(V_1, d_{V_1})$ to $(X, d)$ (Theorem 3.10). Similarly $f_n$ is continuous from $(V_n, d_{V_n})$ to $(V_{n-1}, d_{V_{n-1}})$ for every $n > 1$.
(3) Since $U_n$ is dense in $(X, d)$, $V_n$ is also dense in $(X, d)$ for every $n \in \mathbb{N}$ (Theorem 2.12). Therefore $f(V_1)$ is dense in $(X, d)$. For $n > 1$, $V_n$ is dense in $(V_{n-1}, d)$ (Theorem 2.15). Since $d_{V_{n-1}}$ is equivalent to $d$, $V_n$ is dense in $(V_{n-1}, d_{V_{n-1}})$ (Theorem 3.10), and, therefore, $f_n(V_n)$ is dense in $(V_{n-1}, d_{V_{n-1}})$.

Therefore $\bigcap_{n \in \mathbb{N}} (f_1 \circ \cdots \circ f_n)(V_n)$ is dense in $(X, d)$ (Theorem 4.5). □

Baire's Theorem is very popular in functional analysis. For instance, it can be used to prove the Open Mapping Theorem and the Uniform Boundedness Principle.

# Bibliography

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