Senior Team Solutions - TD Canada Trust High School Mathematics Competition 2011

1. We have $a_2 = 135$, $a_3 = 153$, and thereafter, every $a_n = 153$, so $a_{2011} = 153$.
2. First, we may assume that the vertices of the triangle lie on the rectangle. If not, then we shrink the rectangle both horizontally and vertically until the vertices lie on it. Doing so does not affect the triangle, but reduces the perimeter of the rectangle, thus making our conclusion stronger. Now, if the triangle is ABC, then to get the perimeter, we travel from A to B, B to C, then C to A, in each case using a straight line. But we can also calculate the perimeter of the rectangle by going from A to B, B to C, C to A, following instead the rectangle. But now we aren't following straight lines anymore (at least, not in all 3 parts). Since the shortest distance between 2 points is a straight line, the perimeter of the triangle is smaller.
3. Given three consecutive positive integers, one must be divisible by 3. Apply this to $2^n - 1$, 2^n and $2^n + 1$. Evidently 3 is not a factor of 2^n, so one of the others is divisible by 3. If it is prime, then it must equal 3. So either $2^n - 1 = 3$, and $n = 2$, or $2^n + 1 = 3$, and $n = 1$. But we are assuming that $n > 2$.
4. Notice that $a_{n+2} + a_{n+1} = 2(a_{n+1} + a_n)$ for all $n \geq 1$. That is, increasing n by 1 doubles the size of $a_{n+1} + a_n$. It follows that $a_{1000} + a_{999} = 2^{998}(a_2 + a_1) = 3 \cdot 2^{998}$.
5. Squaring both sides, we get $x^{\log_{10} x} = 100$. Now taking the base 10 log of each side, we get $(\log_{10} x)^2 = 2$. Thus, $\log_{10} x = \pm\sqrt{2}$, so $x = 10^{\pm\sqrt{2}}$.
6. Observe that
$$12^3 = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = (x^3 + y^3) + 3(x+y)(xy) = 36 + 3(12)(xy).$$
Thus, $xy = (12^3 - 36)/36 = 47$. But also,
$$36 = x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 12(x^2 + y^2 - 47).$$
That is, $x^2 + y^2 = (36/12) + 47 = 50$.
7. As the ant turns 60 degrees each time, the internal angle of the shape traced out is $180 - 60 = 120$. Since the ant moves the same distance before each turn, the shape traced out is a regular hexagon. The total distance is therefore 6 units.
8. We have $2^x(2^2 + 1) = 5^y(5 - 1)$. That is, $5(2^x) = 4(5^y)$. In other words, $2^{x-2} = 5^{y-1}$. But an integer power of 2 cannot equal an integer power of 5 unless both are 1. Therefore, the exponents are zero. That is, $x = 2$, $y = 1$.
9. Solution 1: The centres of the first three circles are the vertices of an equilateral triangle with side 2. The distance from the centre of the fourth circle to any of the other centres (that is, the radius of the fourth circle) is 2/3 of the height, namely $(2/3)(2)(\sqrt{3}/2) = 2\sqrt{3}/3$.

   Solution 2: For the sake of convenience, we may as well place the centres of two of the circles at $(-1, 0)$ and $(1, 0)$. We can see that the centre of the third must be on the y-axis, say $(0, a)$. But its distance from the other two centres is 2, so $\sqrt{(-1-0)^2 + (0-a)^2} = 2$. That is, $a = \pm\sqrt{3}$. We may as well use $a = \sqrt{3}$, so the third centre is $(0, \sqrt{3})$. It is also clear that the centre of the fourth circle, being equidistant from the first two centres, must be on the y-axis as well, say $(0, b)$. But then $(0, b)$ must be equidistant from $(1, 0)$ and $(0, \sqrt{3})$. It now follows that $\sqrt{(1-0)^2 + (0-b)^2} = \sqrt{(0-0)^2 + (b-\sqrt{3})^2}$. Squaring both sides and solving for b, we get $b = \frac{1}{\sqrt{3}}$. Thus, the radius is the distance from $(0, b)$ to $(1, 0)$, which is $\sqrt{(0-1)^2 + (\frac{1}{\sqrt{3}} - 0)^2} = \frac{2}{\sqrt{3}}$.

10. Let r be the number of red chameleons, b the number of blue and g the number of green, at any given moment. Let $x = r - b$, $y = b - g$ and $z = g - r$. Let us consider the meeting of a red and a blue. Then r and b each drop by one, but g increases by 2. We see that x doesn't change, y decreases by 3 and z increases by 3. The other possible meetings are similar: one of $\{x, y, z\}$ stays the same, and the other 2 change by ± 3. If all chameleons ever become the same colour, then two of $\{r, b, g\}$ must be 0. In particular, one of $\{x, y, z\}$ will be zero. But if x, y and z only ever change by 3 at a time, then the only way we can get to 0 is if one of them is a multiple of 3 to begin with. However, we start with $x = -7$, $y = -10$, $z = 17$, and none of these are multiples of 3. Therefore, it is impossible.