Senior Team Solutions - TD Canada Trust High School Mathematics Competition 2010

1. We have $2 = \log_5 x - \log_5 y = \log_5(x/y)$, hence $x/y = 5^2 = 25$.
2. We have $3 = m^2$ and $k = m^3$. Therefore $mk = mm^3 = m^4 = 3^2 = 9$.
3. Notice that $(1+1/a)(1+1/b) = ((a+1)/a)((b+1)/b) = (ab+a+b+1)/ab$. But $ab/ab = 1$ (and can be ignored), and $a+b = 6$, so we want to minimize $7/ab$. This means maximizing ab. But $b = 6-a$, so we maximize $a(6-a) = 9-(3-a)^2$. As $(3-a)^2 \geq 0$, the maximum occurs when $a = 3$ and, therefore, $b = 3$. The minimum value is $(1+1/3)(1+1/3) = 16/9$.
4. Solution (i): By construction, BE and DF have the same length and, therefore, EC and FC have the same length. Since ECF is a right triangle and EF has length 10, we see that the length of EC is $10/\sqrt{2} = 5\sqrt{2}$. Let x be the side length of the square. Then considering ABE, we note that BE has length $x - 5\sqrt{2}$ and, therefore, $x^2 + (x-5\sqrt{2})^2 = 10^2$. Using the quadratic equation (and discarding the negative solution), we get $x = 5(\sqrt{2}+\sqrt{6})/2$.

   Solution (ii): Let x be the side length of the square, and let the length of BE (and also DF) be a. Then from triangle ABE, we see that $x^2 + a^2 = 100$. From triangle EFC, we get $(x-a)^2 + (x-a)^2 = 10^2$, hence $2x^2 - 4ax + 2a^2 = 100$, or $x^2 - 2ax + a^2 = 50$. Subtracting one equation from the other, we get $2ax = 50$. Thus, $a = 25/x$. But then $x^2 + (25/x)^2 = 100$, and multiplying through by x^2, we get $x^4 - 100x^2 + 625 = 0$. Applying the quadratic equation to solve for x^2, we get $x^2 = 50 \pm 25\sqrt{3}$. Now, $50 - 25\sqrt{3} < 25$, so if this were the answer, then we would have $a = 25/x > 5$, and therefore $x < a$, which is impossible. Thus, $x^2 = 50 + 25\sqrt{3}$, hence $x = \sqrt{50+25\sqrt{3}} = 5\sqrt{2+\sqrt{3}}$.
5. For both of the two points (x, y), we know that $cx^2 + 19x = x(cx+19)$ is prime. Since x is positive, so is $cx+19$, and we are left with two possibilities: (i) $x = 1$ and $cx + 19 = c + 19$ is prime, or (ii) $cx + 19 = 1$ and x is prime. As c is fixed, each of these yields at most one x-value, so both of these must work. That is, $c + 19$ is prime, and for some prime x, $cx + 19 = 1$. But then $cx = -18$. As the only primes dividing 18 are 2 and 3, we must have $x = 2$ or 3, hence $c = -9$ or -6. But $c + 19$ is prime, hence $c = -9$ doesn't work. Thus, $c = -6$.
6. We have $(25!)^3 - (24!)^3 = (24!)^3(25^3 - 1)$. Now, the prime divisors of 24! are the prime divisors of the numbers from 1 to 24, and certainly the largest of these is 23. On the other hand $25^3 - 1 = 15624 = 2^3 \cdot 3^2 \cdot 7 \cdot 31$. Thus, the largest prime factor is 31.
7. Let a be any root of f. Then $0 = f(1+(a-1)) = f(1-(a-1)) = f(2-a)$. Therefore, $2-a$ is also a root, and we can pair off all of the roots in this way. (We do have to be a bit cautious if $a = 2-a$; that is, if $a = 1$. But since the number of roots is even, this is impossible; the roots must occur in pairs. If we allow for multiple roots, then the multiplicity of 1 must be even; in this case, we get a sum of 2 for each pair anyway.) Thus, we have four pairs of roots, with each pair summing to 2. Therefore, the sum is 8.
8. No matter how many pirates there are, the leader will receive all of the money and everybody else will get nothing. Let us explain why. Certainly, if the leader believes he can get away with this, then his greed will force him to do it, so it is simply a matter of whether he can win the vote. If there are only 2 pirates, then he can clearly do whatever he wants. Number 2 can vote against him, but a tie vote is a victory, so the opinion of number 2 doesn't matter. Suppose we know that our answer is correct if there are n pirates, and consider the case where there are $n+1$ pirates. Again, the leader proposes to give himself everything. Pirate number 2 will vote against him; if this proposal gets voted down, then number 2 becomes the leader of a band of n pirates, and then he knows that he will get everything. What of the remaining pirates? If they accept the plan, they get nothing. What if they vote it down? In that case, they will still not be the leader, and we already know that in a band of n pirates, anybody who is not the leader gets nothing. So the other pirates get nothing anyway. According to the rules, all of the pirates other than number 2 will therefore vote in favour of the plan, thus the plan is adopted.

9. Suppose $n = 10a + b$, with $0 \le b \le 9$. (That is, b is the remainder when n is divided by 10.) The total of amount of money is $n^2 = 100a^2 + 20ab + b^2$. Now, the number of \$10 bills is odd. But $100a^2 + 20ab$ is a multiple of 20; thus, the odd number of \$10 bills comes from b^2. But considering the cases $b = 0, 1, \dots, 9$, we see that the only cases that result in an odd number of bills are $b = 4$ or 6. Thus, the last digit of n^2, which equals the number of loonies, is 6 either way. Therefore, June gets one more \$10 bill than Bob, but Bob gets \$6 in loonies, so June is \$4 ahead. Therefore, she should pay Bob \$2 to make them even.
10. Solution (i): Put the center of the large circle at the origin with the center of the smaller circle at $(-2r, 0)$. The equation of the smaller circle is $(x - (-2r))^2 + y^2 = r^2$, and of the bigger circle is $x^2 + y^2 = (2r)^2$. Their intersection is at points $(x, y) = ((-7/4)r, \pm\sqrt{15}r/4)$. Since A=$(-r, 0)$, line segment OA has length $(\sqrt{6}/2)r$, and so the area of the circle with radius OA is $3\pi r^2/2$.

    Solution (ii): Let B be the centre of the large circle, and C the centre of the small one. Drop a perpendicular from O to CB, meeting CB at D. Let CD $= x$. Then considering the right triangle CDO, we get DO$^2 = r^2 - x^2$. Furthermore, DO^2+BD^2 = OB^2 implies $(r^2 - x^2) + (2r - x)^2 = (2r)^2$. Thus, $x = r/4$, and hence AO^2 = OD^2+AD^2 $= (r^2 - (r/4)^2) + (r - r/4)^2 = 3r^2/2$. Therefore, the area is $3\pi r^2/2$.

    Solution (iii): Define B and C as in the preceding solution. Write θ for angle OCA. Then applying the cosine law to triangle OCB, we get $(2r)^2 = r^2 + (2r)^2 - 2(r)(2r)\cos\theta$. Thus, $\cos\theta = 1/4$. Now applying the cosine law to triangle OAC, we get OA$^2 = r^2 + r^2 - 2(r)(r)(1/4) = 3r^2/2$. Therefore, the area is $3\pi r^2/2$.