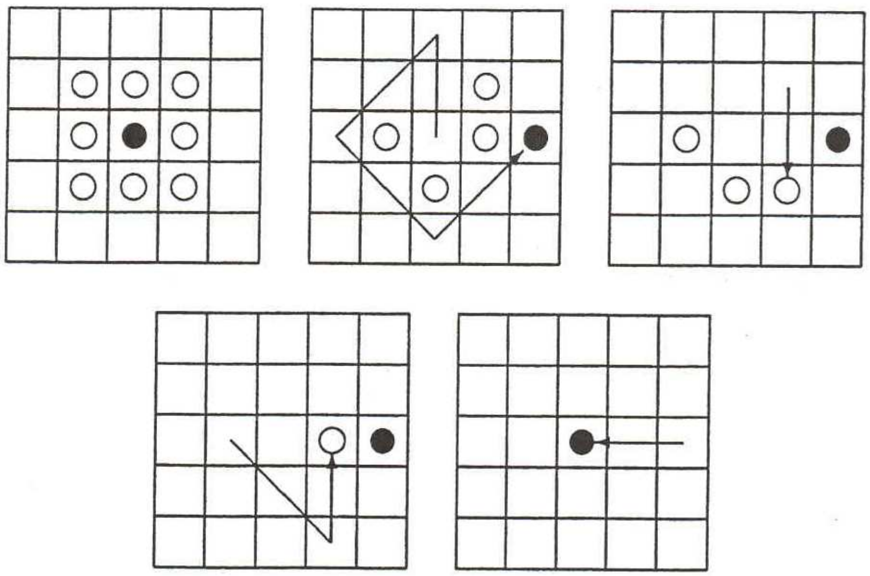
TD Canada Trust Northwestern Ontario High School Mathematics Competition

Senior Team Solutions

1. This is a counting argument. The number of ways to choose 3 people out of 5 is $5(4)(3)/(3(2)(1)) = 10$. We must choose both students, and there are 3 choices for the one teacher, so the probability is 3/10, or 30%.

2. You can do this in 4 jumps. The trick is to note that you first want to use the middle black counter to remove 4 pieces (in a single jump). The last jump puts the black counter back in the middle.



3. After completing n loops of the spiral (counting 1 as loop zero), we will have a $(2n+1)\times(2n+1)$ grid, with the last entry, $(2n+1)^2$, in the lower left corner. The lower right corner is therefore $(2n+1)^2 - 2n$, and the middle of the rightmost column would be $(2n+1)^2 - 2n - n = 4n^2 + n + 1$. So 1, 6 and 19 correspond to $n = 0$, 1, and 2 respectively. The next three numbers are 40, 69 and 106. [Note: The general solution will be depend on where you start your numbering, so $4(n-1)^2 + (n-1) + 1$ would be fine too.]

4. There are only 12 cards with even numbers to go around, and 13 students, so at least one student will have to get 2 odd numbers. Adding them together gives an even number, and an even number multiplied by anything else is even.

5. Let a, b, c and d be the four numbers. So $a \geq \frac{b+c}{2}$, i.e. $2a \geq b+c$. Similarly, $2b \geq a+d$, and $2c \geq a+d$. So $4a \geq 2b+2c \geq 2a+2d$, i.e. $a \geq d$. But $2d \geq b+c$, so $4d \geq 2b+2c \geq 2a+2d$. So $d \geq a$, i.e., $d = a$. We can now show that the other numbers are equal in a similar fashion.)

6. Proceed as we would to find the digits in a positive base. $-2158 = (-2)(1079) + 0$, so the last digit is 0. Then $1079 = (-2)(-539) + 1$, so the next to last digit is 1. Next, $-539 = (-2)(270) + 1$, $270 = (-2)(-135) + 0$, $-135 = (-2)(68) + 1$, $68 = (-2)(-34) + 0$, $-34 = (-2)(17) + 0$, $17 = (-2)(-8) + 1$, and $-8 = (-2)^3$. So the representation is 100010010110.

7. If such integers exist, then $n = (a-b)(a+b)$. Since $a-b \geq 2$ and $a+b \geq a-b$, n is composite. Conversely, if n is composite, say $n = cd$, with $1 < d \leq c < n$, then let $b = (c-d)/2$, $a = (c+d)/2$. We get $n = a^2 - b^2$, and clearly $b \geq 0$, and $a = b + d \geq b + 1$, as required.

8. With k lines in place, the next line will meet each of the other lines exactly once. Each time it crosses a line, it enters a new region, and divides it into two regions, thus adding one new region. So, line $k+1$ adds $k+1$ new regions (as the k lines separate $k+1$ regions). Thus, as the first line gives us 2 regions, we get $2 + 2 + 3 + 4 + \cdots + k = 1 + k(k+1)/2$.

9. Each term describes the preceding term in words. The first entry is "1 one", so we write "11". This is "2 ones", so we write "21". Then "1 two and 1 one", so "1211", and so on. The next three terms are 31131211131221, 13211311123113112211, 11131221133112132113212221.

10. For (i), draw AC to create two triangles. Their areas are certainly no more than $(ab)/2$ and $(cd)/2$ respectively, so the area of the quadrilateral is at most $(ab + cd)/2$, as required.

For (ii), draw a perpendicular bisector to BD, then reflect C in that line to obtain C'. Then ABC'D has the same area as ABCD, and the same side lengths, but in a different order. Use (i).