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Lakehead UNIVERSITY

Department of Mathematical Sciences

LAKEHEAD UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION
April 30, 2008

SENIOR TEAM COMPETITION
Grades 11 and 12

School Name: ______________________________

Team Members: #1 ______________________________ PH: ____________________

#2 ______________________________ PH: ____________________

#3 ______________________________ PH: ____________________

Question #	For Markers Use only
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
GRAND TOTAL	/100

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1. A group consists of three teachers and two students. Three people are selected at random from the group. What is the probability that both of the students are chosen?

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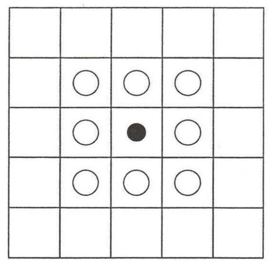
2. On a 5×5 chessboard, place one black counter in the middle of the board, and put eight white counters in the spaces surrounding the black counter. Each counter may jump over an adjacent counter (horizontally, vertically or diagonally) provided that the square immediately beyond is vacant. The jumping counter will land on that square, while the jumped counter is removed (note that the colour doesn't mater; a white counter jumped over a white counter will remove the white counter). Remove all eight white counters this way, while returning the black counter to its initial position. Use the smallest possible number of jumps, where a sequence of consecutive jumps by the same counter counts as a single jump. (In your solution, draw the board after each move. As long as your solution uses the smallest possible number of moves, you don't need to prove this.)



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3. In the number spiral below, give the next three numbers to the right of 11, 2, 1, 6, 19.

13	14	15	16	17			
12	3	4	5	18			
11	2	1	6	19	?	?	?
10	9	8	7	20			
		...	22	21			

Now give a general formula for the numbers in the sequence $1, 6, 19, \ldots$ that appear as part of the middle row, as shown.

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4. A half deck of cards consists of 26 cards, each labeled with an integer 1 through 13. There are two cards labeled with a 1, two cards labeled with a 2, and so on. In a particular math class, there are 13 students. The teacher deals each student two cards randomly from the deck. The students are then asked to add their two numbers together. Then, all 13 of the added numbers are multiplied together. The students will have to do their math homework if the number formed by multiplying all the numbers together is even, and they will not have to do their homework if the number is odd. Explain why the students will always have to do their homework.

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5. Four positive integers are placed at the vertices of a rectangle. Each number is at least as large as the average of the two numbers at the adjacent vertices. Prove that all four numbers are equal.

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6. We know that the number 10 can be used as a base for all positive integers, since we can write each positive integer in a unique way in the form

$$a_0 + a_1(10) + a_2(10^2) + \cdots + a_k(10^k),$$

for some $k \geq 0$, and with each $a_i \in \{0, 1, 2, \ldots, 9\}$. Similarly, we can use -2 as a base for all integers, taking the $a_i \in \{0, 1\}$. For example,

$$-3 = 1 + 0(-2) + 1(-2)^2 + 1(-2)^3.$$

Find a base -2 representation for the base 10 number -2158.

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7. Let n be an odd integer, $n \geq 3$. Show that n is composite (that is, $n = cd$ with $1 < c < n$ and $1 < d < n$) if and only if there exist integers a and b with $a > b + 1 > 0$ and $n = a^2 - b^2$.

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8. Given n lines in the plane, with no two parallel and no three concurrent (that is, no three passing through the same point), into how many regions do they divide the plane? Provide a proof, not just a sketch.

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9. What are the next three terms in the following sequence that starts:

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, \ldots$$

Explain your rule.

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10. Consider a convex quadrilateral ABCD, with side lengths a, b, c and d, as indicated in the diagram. Let s be its area. Show that (i) $2s \leq ab + cd$, and (ii) $2s \leq ac + bd$. (Hint: You can use (i) to prove (ii), but (ii) is tougher.)