2005 Thunder Bay High School Mathematics Competition

Senior Team Answers

For complete solutions to any of the problems or for discussions of any of the problems, email ryanholm72@hotmail.com.

Note to markers: For the full solution questions students are expected to provide much more detail than I have here to receive full marks; this is just a reference sheet.

Note to students: The solutions provided here are only *sketch solutions*. Steps have been omitted and the solutions are meant to guide any student who has made a valiant attempt at solving the problem.

Full Solution (Answers and sketch solutions):

1. 15. Let x be the original number. Then $\frac{x-9}{3}=43$, which gives that x = 138. So the correct answer is $\frac{138-3}{9}=15$.

2. 2. Since 2A3 + 326 = 5B9, we have A + 2 = B. Since 5B9 is divisible by 9, the sum of its digits must be divisible by 9. So we must have B = 4 and so A = 2.

3. (c) is necessarily true because I tells us that there exists students who are not honest and II tells us that such students cannot be math club members. (a) and (b) may not be true because each requires that the set of all math club members be nonempty, which is not required by I or II. Again, I and II would allow the set of all math club members to be a nonempty subset of the set of all students, but neither (d) or (e) permits this and accordingly may be not true.

4. 1004. Finding $F(n)$ for $n=2,3,4,5$ quickly reveals the pattern that $F(n)=\frac{n+3}{2}$.

5. The lowest common multiple of 5, 8, 12, and 18 is 360. So every 360 minutes (6 hours) the buses will all be together at the terminal again. So this will happen twice before 10 pm.

6. Let $a=x^5$ and $b=y^5$. Notice that since x > y, then we must have a > b. Substituting in a and b into the system and solving for a and b gives $a=1$ and $b=\frac{1}{2}$. Thus, $x=1$ and $y=\frac{1}{\sqrt[5]{2}}$.

7. 162. Since each match of the tournament eliminates one and only one competitor, no matter how we set up the tournament it will always take exactly 162 games to declare a winner. This is because in each of 162 matches there will be a loser so after 162 matches only 1 person will remain without a loss.

8. 1. $$S = \frac{1}{(\log_2 (1)(2)...(n))} + \frac{1}{(\log_3 (1)(2)...(n))} + ... + \frac{1}{(\log_n (1)(2)...(n))}$$

$$= \frac{\log_{10} 2}{(\log_{10} (1)(2)...(n))} + \frac{\log_{10} 3}{(\log_{10} (1)(2)...(n))} + ... + \frac{\log_{10} 4}{(\log_n (1)(2)...(n))}$$

$$= \frac{\log_{10} (2)(3)...(n)}{(\log_{10} (1)(2)...(n))}$$
$$= 1$$

9. $7^{\frac{1}{2}} + 7^{\frac{1}{3}} + 7^{\frac{1}{4}} < 9^{\frac{1}{2}} + 8^{\frac{1}{3}} + 16^{\frac{1}{4}} = 3 + 2 + 2 = 7$. The second statement is not true since in fact $4^{\frac{1}{2}} + 4^{\frac{1}{3}} + 4^{\frac{1}{4}} > 4^{\frac{1}{2}} + 1^{\frac{1}{3}} + 1^{\frac{1}{4}} = 2 + 1 + 1 = 4$.

10. 3. It helps to draw a picture of a regular tetrahedron. It is then intuitive that 3 pairs of edges are perpendicular. To show this more rigorously, notice that in space a straight line that is perpendicular to a plane is perpendicular to all straight lines in the same plane. Consider any edge, call it E, of the regular tetrahedron. Notice that the only edge that could be perpendicular to E is the one that does not share a common vertex with E, call it F, since the ones that share a common vertex with E all meet E at 60 degrees since all the faces are equilateral triangles. We will show that F is in fact perpendicular to it, and hence it follows that there are 3 such pairs since there are 6 edges in a regular tetrahedron. Consider the set of all points that are of equal distance from the two vertices of E. This is the plane, call it P, which is perpendicular E. Since each of the two vertices of F are of equal distances from the two vertices of E, F lies in the plane P. Hence F is perpendicular to E.

11. Let x be the amount of water present when the pumping begins, y the amount leaking per hour, and z the amount that each person can remove per hour. Then the information in the problem gives us the two equations $x + 3y = 36z$ and $x + 10y = 50z$, since each side of each equation is an expression for the total amount of water removed from the boat. We can solve these two equations for x and y to get that $x = 30z$ and $y = 2z$. We want to find the number n such that $x + 2y = 2nz$. Well, substituting in our expressions for x and y in terms of z gives us that $30z + 4z = 2nz$ and thus $n = 17$.

12. Notice 9 divides 1854. So using the hint, B, C, and D will all be divisible by 9. Notice that $A < 10000^{2000} = 10^{8000}$. So A has at most 8000 digits. If all these digits were 9 then B would be as large as possible. In this case $B \leq 9 \times 8000 = 72000$. Thus B has at most 5 digits and if all these digits were 9 then C would be as large as possible. In this case $C \leq 9 \times 5 = 45$. So C is some integer between 1 and 45 and $C = 39$ gives the largest possible value for D of 12 so $D \leq 12$. But by the second line of this solution we know that 9 divides D. Since D is a positive integer it must be 9.