Senior Individual Solutions - TD Canada Trust High School Mathematics Competition 2011

Multiple Choice Questions.

1. B.
2. C.
3. D.
4. B.
5. B.
6. C.
7. D.
8. A.
9. B.
10. D.
11. C.
12. B.
13. E.
14. B.
15. B.

Full Solution.

1. If $2n+1$ divides $n^3 - n + 3$, then it also divides $8(n^3 - n + 3)$. But performing polynomial long division, $8(n^3 - n + 3) = (2n+1)(4n^2 - 2n - 3) + 27$. Thus, $2n+1$ divides 27. Since n is positive, it can only be 1, 4 or 13. We find that, in fact, each of these values do work, so these are our three solutions.
2. (i) We have $(x + 1/x)^2 = x^2 + 1/x^2 + 2 = 6$. Since x is positive, $x + 1/x = \sqrt{6}$. (ii) Notice that $(x+1/x)^3 = x^3 + 1/x^3 + 3(x+1/x)$. Now, from (i), $x + 1/x = \sqrt{6}$, so $x^3 + 1/x^3 = (\sqrt{6})^3 - 3\sqrt{6} = 6\sqrt{6} - 3\sqrt{6} = 3\sqrt{6}$. (iii) Here, $(x + 1/x)^5 = x^5 + 1/x^5 + 5(x^3 + 1/x^3) + 10(x + 1/x)$. Using (i) and (ii), we get $x^5 + 1/x^5 = (\sqrt{6})^5 - 5(3\sqrt{6}) - 10\sqrt{6} = 36\sqrt{6} - 15\sqrt{6} - 10\sqrt{6} = 11\sqrt{6}$.
3. If $n = 1$, then $2^n - 1 = 1$, which is not prime. Thus, take $n > 1$ and for the purposes of contradiction, assume that n is composite, say $n = ab$, where $a > 1$ and $b > 1$. Then
$$2^{ab} - 1 = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1)$$
gives a factorization of $2^n - 1$. Since $1 < 2^a - 1 < 2^{ab} - 1$, we see that $2^n - 1$ is not prime, giving us a contradiction.