Multiple Choice Questions.

1. A.
2. C.
3. D.
4. B.
5. B.
6. D.
7. E.
8. E.
9. A.
10. C.
11. C.
12. B.
13. A.
14. This is equivalent to $(2\log_a(c) - \log_b(c))(2\log_a(c) - 3\log_b(c)) = 0$. So then $\log_a(c^2) = \log_b(c)$ or $\log_b(c^3)$. If we let $d = \log_a(c^2)$, then $a^d = c^2$ and $b^d = c$ or c^3. That is, $b^{2d} = a^d$ or a^{3d}. Since $d \neq 0$, we get $b^2 = a$ or a^3. The answer is (E).
15. C.

Full Solution.

1. The answer is 108. Let y denote $\angle GEF = \angle DEC$ and z denote $\angle DCE = \angle ACB$. We then have three equations, with three unknowns, that is
$$\begin{aligned} 4x + 3x + y &= 180 \\ 5x + y + z &= 180 \\ 6x + 2x + z &= 180. \end{aligned}$$
We solve for x, y and z. The first and last equation give $y = 180 - 7x$ and $z = 180 - 8x$, respectively. Subbing into the second equation gives
$$5x + 180 - 7x + 180 - 8x = 180 \iff 10x = 180.$$
So $x = 18$, thus $\angle CAB = 6 \cdot 18 = 108$.
2. The answer is 2009. You can prove this by induction, or you can notice the pattern for the first couple of terms. For example
$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} = \frac{5}{8} = \frac{5}{2 \cdot 4}$$
3. Because $a_1 + \cdots + a_n$ is even, there are an even number of odd numbers among $a_1, \ldots, a_n$. If there is an integer x such that $p(x) = 0$, then $1 = -(a_1 x + \cdots + a_n x^n)$. Now, x cannot be even since the right hand side would be even, but the left hand side would be odd. On the other hand, if x is odd, then $x, x^2, x^3, \ldots, x^n$ are all odd. If a_i is even, then $a_i x^i$ is even. If a_i is odd, then $a_i x^i$ is odd. But since there are an even number of a_i's with a_i odd, we get an even number of $a_i x^i$'s that are odd. So, when we sum the right hand side, we get an even number, but the left hand side is odd. Thus, there is no solution.
4. Let E be the vertex on BC and F the vertex on CD. Let x be the length of BE which also equals the length of DF. Then the area of the equilateral triangle is 1 minus the sum of the areas of ADF, ECF and ABE, which are $x/2$, $(1-x)^2/2$, and $x/2$ respectively, so we get $(1-x^2)/2$. If y is the side length of the equilateral triangle, then using the right triangles, we get $y^2 = 1 + x^2 = 2(1-x)^2$, so $x^2 - 4x + 1 = 0$. One of the roots of this polynomial is larger than 1, so we want $x = 2 - \sqrt{3}$, and the area is therefore $2\sqrt{3} - 3$.

5. Let A' be the intersection point of the circles with centers at B and C, so that, by symmetry, $A'B'C'$ and ABC are both equilateral triangles. Again, by symmetry, $\triangle ABC$ and $\triangle A'B'C'$ have a common centroid; call it K. Let M be the midpoint of the line segment BC. From the triangle $A'BC$ we see that the length $A'M = \sqrt{r^2 - 1}$. Since the corresponding lengths in similar triangles are proportional,

$$\frac{B'C'}{BC} = \frac{A'K}{AK}.$$

Since the equilateral $\triangle ABC$ has sides of length 2, we get that $B'C' = \frac{A'K}{AK}$ and also that altitude AM has length $\sqrt{3}$. Consequently, $AK = \frac{2}{3}AM = \frac{2}{3}\sqrt{3}$ and $MK = \frac{1}{3}AM = \frac{1}{3}\sqrt{3}$. So

$$A'K = A'M + MK = \sqrt{r^2 - 1} + \frac{\sqrt{3}}{3}.$$

Thus

$$B'C' = 2\frac{\sqrt{r^2 - 1} + \sqrt{3}/3}{2(\sqrt{3}/3)} = \sqrt{3(r^2 - 1)} + 1.$$