Senior Team - Solutions:

1.

$3 = ((4+4+4) \div 4$

$4 = \sqrt{4+4+4+4}$

$5 = (4 \times 4 + 4) \div 4)$

$6 = 4! \times 4 \div (4 \times 4)$

$7 = 4 + 4 - (4 \div 4)$

$8 = 4+4+4-4$

$9 = 4+4+(4 \div 4)$

$10 = (44 - 4) \div 4$

$11 = 44 \div \sqrt{4 \times 4}$

$12 = 4 \times 4 - \sqrt{4 \times 4}$

2. For each n, $1^n = 1$ and 4^n is divisible by 4, so we get a remainder of 1 for those two terms. When $n = 1$, we get 10, which has a remainder of 2. Otherwise, 4 divides 2^n, so we have only to examine 3^n. But here we have $3^n = (4 - 1)^n = 4^n - ... + (-1)^n$, and each term except the last is divisible by 4, so we get a remainder of $(-1)^n$ here, giving a total remainder of 2 when n is even, and zero when n is odd.

3. Let x be the number thought of by the person who announced 4. Then if a is the number thought of by the person who announced 2, then the average of x and a must be 3, so $a = 6- x$. If b is the number thought of by the person who announced 5, then the average of a and b is 1, so $b = 2 - a = x - 4$. If c is the number thought of by the person who announced 3, then the average of b and c is 4, so $c = 8 - b = 12 - x$. If d is the number thought of by the person who announced 1, then the average of c and d is 2, so $d = 4 - c = x - 8$. But also, the average of x and d is 5, so $d = 10 - x$. Thus, $x-8 = 10 - x$, and $x = 9$.

4. Let a be the length of any side of the hexagon. Joining diagonally opposite vertices in the hexagon, we can divide it into 6 equilateral triangles with side length a. Now, since the triangle has the same perimeter, each side has length $2a$. Joining the midpoints of the three sides together, we divide the triangle into 4 equilateral triangles with side length a. Thus, the hexagon has area equal to 6/4 times that of the triangle, or 15.

5. The first equation can be rewritten as $y = \log(x^5)$. Taking 10 to the power of both sides, we have $x^5 = 5x$, so $x = 0$ (which is not allowed), or $x^4 = 5$. As x must be positive, there is just one solution.

6. Notice that $a_{n+1} = a_n - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2}.$ Since $\frac{1}{n+1} = \frac{2}{2n+2} < \frac{1}{2n+1} + \frac{1}{2n+2}$, we have our first inequality. For the second, notice that a_n is a sum of n terms, each of which is smaller than $\frac{1}{n}$.

7. If a palindrome has digits $abba$, then notice that the number with digits $cddc$ is also a palindrome, where $c = 10 - a$ and $d = 9 - b$. Furthermore, it is a different palindrome, since $b + d = 9$ implies that $b \neq d$. Thus we can pair up all of the palindromes in this way, and notice that any such pair adds up to 11 000. There are 9 choices for a and 10 for b, thus 90 palindromes and 45 pairs. So the sum is 45(11000) = 495000.

8. Since $\cos(2x) = \cos^2(x) - \sin^2(x)$, this is $\sum_{n=1}^{2007} n\cos(n\pi)$. Now, $\cos(n\pi)$ is 1 when n is even and -1 when n is odd, so this is -1 +2 - 3 + 4 -5 + . . . - 2007, and taking two numbers at a time, we get 1 + 1 + 1 + . . .+1 -2007 = 1003 - 2007 = -1004.

9. Partition the cube into 8 equal sub-cubes. One of them must contain at least 2 of the 9 points by the

pigeonhole principle, so the maximum possible distance between them is the distance between diagonally opposite vertices, which is $\sqrt{3(1/2)^2} = \sqrt{3}/2$, as required. To show that no smaller number will suffice, use the 8 vertices and the centre of the cube.

10. For convenience, let P be the origin, and say the circle is $(x - a)^2 + (y - b)^2 = r^2$. If (x, y) is any point on the circle, then the midpoint in question is $(x/2, y/2)$. Notice that $(x/2 - a/2)^2 + (y/2 - b/2)^2 = 1/4\,((x-a)^2 + (y-b)^2) = (r/2)^2$, so the midpoints lie on a circle centered at $(a/2, b/2)$ with radius $r/2$.