An Introduction to Compact Operators

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Abstract

The purpose of this project is to first review some concepts from Functional Analysis and Operator Algebra, then to apply these concepts to an in depth introduction to Compact Operators and the Spectra of Compact Operators, leading to The Fredholm Alternative. Topics discussed include Normed Spaces, Hilbert Spaces, Linear Operators, Bounded Linear Operators, and Compact Operators. The main source for the statements of results is [1]. It is recommended that anyone interested in specifics not provided here consult this source for further details.

1 Introductory Material

Definition 1.1. Let V and W be vector spaces over the same scalar field $\mathbb{F}$. A function $T : V \to W$ is called a *linear transformation* (or *mapping*) if, for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in V$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Definition 1.2. The set of all linear transformations $T : V \to W$ will be denoted by $L(V, W)$

Definition 1.3. Let V and W be vector spaces and $T \in L(V, W)$.

1. The *image* of T (often known as the *range* of T) is the subspace $\mathrm{Im}(T) = T(V)$; the *rank* of T is the number $r(T) = \dim(\mathrm{Im}(T))$.

2. The *kernel* of T (often known as the *null-space* of T) is the subspace $\mathrm{Ker}(T) = \{x \in V : T(x) = 0\}$; the *nullity* of T is the number $n(T) = \dim(\mathrm{Ker}(T))$.

Lemma 1.4. *Let V, W be vector spaces and $T \in L(V, W)$.*

1. *T is one-to-one if and only if the equation $T(x) = 0$ has only the solution $x=0$. This is equivalent to $Ker(T) = \{0\}$ or $n(T) = 0$.*

2. *T is onto if and only if $Im(T) = W$. If $dim(W)$ is finite this is equivalent to $r(T)=dim(W)$.*

3. *$T \in L(V, W)$ is bijective if and only if there exists a unique transformation $S \in L(W, V)$ which is bijective and $S \circ T = I_V$ and $T \circ S = I_W$.*

Lemma 1.5. *Let V be a vector space and let $T \in L(V)$. Let $\{\lambda_1, \dots, \lambda_k\}$ be a set of distinct eigenvalues of T, and for each $1 \leq j \leq k$ let x_j be an eigenvector corresponding to λ_j. Then the set $\{x_1, \dots, x_k\}$ is linearly independent.*

Definition 1.6. Let (M, d) be a metric space. For any $x \in M$ and any number $r > 0$, the set

$$B_x(r) = \{y \in M : d(x, y) < r\}$$

will be called the *open ball* with centre x and radius r. The set $\{y \in M : d(x, y) \leq r\}$ will be called the *closed ball* with centre x and radius r.

Definition 1.7. Let (M, d) be a metric space and let $A \subset M$.

1. A is *bounded* if there is a number $b > 0$ such that $d(x, y) < b$ for all $x, y \in A$.

2. A is *open* if, for each point $x \in A$, there is an $\epsilon > 0$ such that $B_x(\epsilon) \subset A$.

3. A is *closed* if the set $M \backslash A$ is open.

4. A point $x \in M$ is a *closure point* of A if, for every $\epsilon > 0$, there is a point $y \in A$ with $d(x, y) < \epsilon$ (equivalently, if there exists a sequence $\{y_n\} \subset A$ such that $y_n \to x$).

5. The *closure* of A, denoted $\bar{A}$ or A^-, is the set of all closure points of A.

6. A is *dense* (in M) if $\bar{A} = M$

Theorem 1.8. *Let (M,d) be a metric space and let $A \subset M$.*

1. *$\bar{A}$ is closed and is equal to the intersection of the collection of all closed subsets of M which contain A (so $\bar{A}$ is the smallest closed set containing A).*

2. *A is closed if and only if $A = \bar{A}$.*

3. *A is closed if and only if, whenever $\{x_n\}$ is a sequence in A which converges to an element $x \in M$, then $x \in A$.*

4. *$x \in \bar{A}$ if and only if $inf\{d(x,y) : y \in A\} = 0$.*

5. *For any $x \in M$ and $r > 0$, the "open" and "closed" balls in Definition 1.6 are open and closed in the sense of Definition 1.7. Furthermore,*

$$\overline{B_x(r)} \subset \{y \in M : d(x,y) \leq r\}$$

but these sets need not be equal in general (however, for most of the spaces we will consider these sets are equal).

6. *A is dense if and only if, for any element $x \in M$ and any number $\epsilon > 0$, there exists a point $y \in A$ with $d(x,y) < \epsilon$ (equivalently, for any element $x \in M$ there exist a sequence $\{y_n\} \subset A$ such that $y_n \to x$).*

Theorem 1.9. ***(Baire's Category Theorem)***
If (M,d) is a complete metric space and $M = \bigcup_{j=1}^{\infty} A_j$, where each $A_j \subset M$, $j = 1, 2, \ldots$, is closed, then at least one of the sets A_j contains an open ball.

Theorem 1.10. ***(Bolzano-Weierstrass Theorem)***
Every closed, bounded subset of $\mathbb{F}^k$ is compact.

Definition 1.11. Let (M,d) be a compact metric space. The set of continuous functions $f : M \to \mathbb{F}$ will be denoted $C_{\mathbb{F}}(M)$. We define a metric on $C_{\mathbb{F}}(M)$ by

$$d(f,g) = sup\{|f(x) - g(x)| : x \in M\}$$

Definition 1.12. Suppose that (M,d) is a compact metric space and $\{f_n\}$ is a sequence in $C(M)$, and let $f : M \to \mathbb{F}$ be a function.

1. $\{f_n\}$ converges *pointwise* to f if $|f_n(x) - f(x)| \to 0$ for all $x \in M$.

2. $\{f_n\}$ converges *uniformly* to f if $sup\{|f_n(x) - f(x)| : x \in M\} \to 0$.

Theorem 1.13. *The metric space C(M) is complete.*

Theorem 1.14. *Suppose that (M,d) is a metric space and $A \subset M$.*

1. *If A is compact then it is separable.*

2. *If A is separable and $B \subset A$ then B is separable.*

2 Normed Spaces

Lemma 2.1. *Let X be a vector space and let $\| \cdot \|_1$, $\| \cdot \|_2$ and $\| \cdot \|_3$ be three norms on X. Let $\| \cdot \|_2$ be equivalent to $\| \cdot \|_1$ and let $\| \cdot \|_3$ be equivalent to $\| \cdot \|_2$.*

1. *$\| \cdot \|_1$ is equivalent to $\| \cdot \|_2$.*

2. *$\| \cdot \|_3$ is equivalent to $\| \cdot \|_1$.*

Lemma 2.2. *Let X be a vector space and let $\| \cdot \|$ and $\| \cdot \|_1$ be norms on X. Let d and d_1 be metrics defined by $d(x, y) =\| x - y \|$ and $d_1(x, y) =\| x - y \|_1$. Suppose that there exists $K > 0$ such that $\| x \|\leq K \| x \|_1$ for all $x \in X$. Let $\{x_n\}$ be a sequence in X.*

1. *If $\{x_n\}$ converges to x in the metric space $(X,\ d_1)$ the $\{x_n\}$ converges to x in the metric space (X,d)*

2. *If $\{x_n\}$ is Cauchy in the metric space (X,d_1) then $\{x_n\}$ is Cauchy in the metric space (X,d).*

Corollary 2.3. *Let X be a vector space and let $\| \cdot \|$ and $\| \cdot \|_1$ be equivalent norms on X. Let d and d_1 be the metrics defined by $d(x, y) =\| x - y \|$ and $d_1(x, y) =\| x - y \|_1$. Let $\{x_n\}$ be a sequence in X.*

1. *$\{x_n\}$ converge to x in the metric space (X,d) if and only if $\{x_n\}$ converges to x in the metric space (X,d_1).*

2. *$\{x_n\}$ is Cauchy in the metric space (X,d) if and only if $\{x_n\}$ is Cauchy in the metric space (X,d_1).*

3. *(X,d) is complete if and only if (X,d_1) is complete.*

Theorem 2.4. *Let X be a finite-dimensional vector space with norm $\| \cdot \|$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis for X. Another norm on X is*

$$\| \sum_{j=1}^{n} \lambda_j e_j \|_1 = \Big(\sum_{j=1}^{n} |\lambda_j|^2 \Big)^{\frac{1}{2}} \tag{2.0.1}$$

The norms $\| \cdot \|$ and $\| \cdot \|_1$ are equivalent.

Corollary 2.5. *If $\| \cdot \|$ and $\| \cdot \|_2$ are any two norms on a finite-dimensional vector space X then they are equivalent.*

Lemma 2.6. *Let X be a finite-dimensional vector space over $\mathbb{F}$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis for X. If $\| \cdot \|_1 : X \to \mathbb{R}$ is the norm on X defined by* (2.0.1) *the X is a complete metric space.*

Corollary 2.7. *If $\| \cdot \|$ is any norm on a finite-dimensional space X then X is a complete metric space*

Corollary 2.8. *If Y is a finite-dimensional subspace of a normed vector space X, then Y is closed.*

Theorem 2.9. *If X is an infinite-dimensional normed vector space then neither $D = \{x \in X :\| x \|\leq 1\}$ nor $K = \{x \in X :\| x \|= 1\}$ is compact.*

Definition 2.10. A *Banach space* is a normed vector space which is complete under the metric associated with the norm.

Theorem 2.11. *Let X be a Banach space and let $\{x_n\}$ be a sequence in X. If the series $\displaystyle\sum_{k=1}^{\infty} \| x_k \|$ converges the the series $\displaystyle\sum_{k=1}^{\infty} x_k$ converges.*

3 Hilbert Spaces

Definition 3.1. Let X be a complex vector space. An *inner product* on X is a function $(\cdot,\cdot) : X \times X \to \mathbb{C}$ such that for all $x, y, z \in X, \alpha, \beta \in \mathbb{C}$,

1. $(x, x) \in \mathbb{R}$ and $(x, x) \geq 0$;
2. $(x, x) = 0$ if and only if $x = 0$;
3. $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$
4. $(x, y) = \overline{(y, x)}$

Lemma 3.2. *Let X be an inner product space, $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$. Then,*

1. *(0,y)=(x,0)=0*
2. $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$
3. $(\alpha x + \beta y, \alpha x + \beta y) = |\alpha|^2(x, x) + \alpha\bar{\beta}(x, y) + \bar{\alpha}\beta(y, x) + |\beta|^2(y, y)$

Lemma 3.3. *Let X be an inner product space and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences in X, with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then*

$$\lim_{n\to\infty}(x_n, y_n) = (x, y)$$

Lemma 3.4. *Let $\{v_1, \dots, v_k\}$ be a linearly independent subset of an inner product space X, and let S=Sp $\{v_1, \dots, v_k\}$. Then there is an orthonormal basis $\{e_1, \dots, e_k\}$ for S.*

Definition 3.5. An inner product space which is complete with respect to the metric associated with the norm induced by the inner product is called a *Hilbert Space.*

Lemma 3.6. *If $\mathcal{H}$ is a Hilbert space and $Y \subset \mathcal{H}$ is a linear subspace, then Y is a Hilbert space if and only if Y is closed in $\mathcal{H}$.*

Definition 3.7. Let X be an inner product space and let A be a subset of X. The *orthogonal complement* of A is the set

$$A^{\perp} = \{x \in X : (x, a) = 0, \forall a \in A\}$$

Lemma 3.8. *If X is an inner product space and $A \subset X$ then:*

1. *If $0 \in A$ then $A \cap A^{\perp} = \{0\}$, otherwise $A \cap A^{\perp} = \varnothing$.*
2. *$A^{\perp}$ is a closed linear subspace of X.*
3. $A \subset (A^{\perp})^{\perp}$

Lemma 3.9. *Let Y be a linear subspace of an inner product space X. Then*

$$x \in Y^{\perp} \iff \| x - y \| \geq \| x \|, \forall y \in Y$$

Theorem 3.10. *Let A be a non-empty, closed, convex subset of a Hilbert space $\mathcal{H}$ and let $p \in \mathcal{H}$. Then there exists a unique $q \in A$ such that*

$$\| p - q \| = inf\{\| p - a \| : a \in A\}$$

Theorem 3.11. *Let Y be a closed linear subspace of a Hilbert space $\mathcal{H}$. For any $x \in \mathcal{H}$, there exists a unique $y \in Y$ and $z \in Y^{\perp}$ such that $x = y + z$. Also, $\| x \|^2 = \| y \|^2 + \| z \|^2$*

Corollary 3.12. *If Y is a closed linear subspace of a Hilbert space $\mathcal{H}$ then $Y^{\perp\perp} = Y$.*

Corollary 3.13. *If Y is any linear subspace of a Hilbert space $\mathcal{H}$ then $Y^{\perp\perp} = \bar{Y}$*

Definition 3.14. Let X be an inner product space. A sequence $\{e_n\} \subset X$ is said to be an *orthonormal sequence* if $\| e_n \| = 1$ for all $n \in \mathbb{N}$, and $(e_m, e_n) = 0$ for all $m, n \in \mathbb{N}$ with $m \neq n$.

Theorem 3.15. *Any infinite-dimensional inner product space X contains an orthonormal sequence*

Theorem 3.16.

1. *Finite dimensional normed vector spaces are separable.*
2. *An infinite-dimensional Hilbert space $\mathcal{H}$ is separable if and only if it has an orthonormal basis.*

Example of a Hilbert Space

The set of all sequences in a field is $\mathbb{C}^N = \{\{x_n\} : x_n \in \mathbb{C}\}$

Which is clearly a vector space if we define addition and scalar multiplication as follows

- $\{x_n\} + \{y_n\} \equiv \{x_n + y_n\}$
- $c\{x_n\} \equiv \{cx_n\}$

For simplicity we write x for $\{x_n\}$ meaning the usual vector notation.

$\ell^2_{\mathbb{C}} \subset \mathbb{C}^N$ consists of all sequences x such that $\displaystyle\sum_n |x_n|^2 < \infty$

$\ell^2_{\mathbb{C}}$ is often abbreviated as ℓ^2.
The inner product on ℓ^2 is $(x, y) = \sum_n x_n \overline{y_n}$
The norm on ℓ^2 is $\| x \| = (\displaystyle\sum_n |x_n|^2)^{1/2}$.

This norm is often called the *counting metric* and is directly analogous to the Euclidean Metric from $\mathbb{R}^N$
ℓ^2 is clearly an inner product space and is clearly complete with respect to the induced norm in the same way as $\mathbb{R}^N$ is. Therefore ℓ^2 is a Hilbert Space (and also a Banach Space).

4 Bounded Linear Operators

Lemma 4.1. *Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear transformation. The following are equivalent:*

1. *T is uniformly continuous;*
2. *T is continuous;*
3. *T is continuous at 0;*
4. *there exists a positive real number k such that $\| T(x) \| \leq k$ whenever $x \in X$ and $\| x \| \leq 1$;*
5. *there exists a positive real number k such that $\| T(x) \| \leq k \| x \|$ for all $x \in X$.*

Definition 4.2. Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear transformation. T is said to be *bounded* if there exists a positive real number k such that $\| T(x) \| \leq k \| x \|$ for all $x \in X$.

Definition 4.3. Let X an Y be normed linear spaces. The set of all continuous linear transformations from X to Y is denoted by $B(X, Y)$. Elements of $B(X, Y)$ are also called *bounder linear operators* or *linear operators*, or sometimes just *operators*

Note
For Linear Operators, boundedness is equivalent to continuity. Apply Lemma 4.1 to the definition of a Bounded Linear Operator.

Lemma 4.4. *Let X and Y be normed spaces. If $\| \cdot \| : B(X, Y) \to \mathbb{R}$ is defined by*

$$\|T\| = sup\{\|T(x)\| : \|x\| \leq 1\}$$

then $\| \cdot \|$ is a norm on $B(X, Y)$

Theorem 4.5. *Let X be a finite-dimensional normed space, let Y be any normed linear space and let $T : X \to Y$ be a linear transformation. Then T is continuous.*

Lemma 4.6. *If X and Y are normed linear spaces and $T : X \to Y$ is a continuous linear transformation then Ker(T) is closed.*

Theorem 4.7. *If X is a normed linear space and Y is a Banach space the B(X,Y) is a Banach space.*

Theorem 4.8. *Let X be a Banach space. If $T \in B(X)$ is an operator with $\| T \| < 1$ the $I - T$ is invertible and the inverse is given by*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Theorem 4.9. ***(Open Mapping Theorem)***
Suppose that X and Y are Banach space and $T \in B(X, Y)$ is surjective. Let

$$L = \{T(x); x \in X \text{ and } \| x \| \leq 1\}$$

with closure $\bar{L}$. Then:

1. *there exists $r > 0$ such that $\{y \in Y : \| y \| \leq r\} \subseteq \bar{L}$;*
2. *$\{y \in Y : \| y \| \leq \frac{r}{2} \subseteq L\}$;*
3. *if, in addition, T is one-to-one then T is invertible.*

Corollary 4.10. ***(Closed Graph Theorem)***
If X and Y are Banach spaces and T is a linear transformation from X into Y such that $\mathcal{G}(T)$, the graph of T, is closed, then T is continuous.

Corollary 4.11. ***(Banach's Isomorphism Theorem)***
If X, Y are Banach spaces and $T \in B(X, Y)$ is bijective, then T is invertible.

Lemma 4.12. *If X and Y are normed linear spaces and $T \in B(X, Y)$ is invertible then, for all $x \in X$,*

$$\| Tx \| \geq \| T^{-1} \|^{-1} \| x \| .$$

Theorem 4.13. ***(Uniform Boundedness Principle)***
Let U, X be Banach spaces. Suppose that S is a non-empty set and, for each $s \in S$, $T_s \in B(U, X)$. If, for each $u \in U$, the set $\{\| T_s(u) \| : s \in S\}$ is bounded then the set $\{\| T_s \| : s \in S\}$ is bounded.

Theorem 4.14. ***(Riesz-Fréchet Theorem)***
Let $\mathcal{H}$ be a Hilbert space and let $f \in \mathcal{H}'$. Then there is a unique $y \in \mathcal{H}$ such that $f(x) = f_y(x) = (x, y)$ for all $x \in \mathcal{H}$. Moreover $\| f \| = \| y \|$.

5 Bounded Linear Operators on Hilbert Spaces

Theorem 5.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $T \in B(\mathcal{H}, \mathcal{K})$. There exists a unique operator $T^* \in B(\mathcal{H}, \mathcal{K})$ such that for all $x \in \mathcal{H}$ and all $y \in \mathcal{K}$*

$$(Tx, y) = (x, T^*y)$$

Definition 5.2. If $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces and $T \in B(\mathcal{H}, \mathcal{K})$ the operator T^* constructed above is called the *adjoint* of T.

Lemma 5.3. *Let, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be complex Hilbert spaces, let R, $S \in B(\mathcal{H}, \mathcal{K})$ and let $T \in B(\mathcal{K}, \mathcal{L})$. Let $\lambda, \mu \in \mathbb{C}$. Then:*

1. $(\mu R + \lambda S)^* = \bar{\mu} R^* + \bar{\lambda} S^*$;
2. $(TR)^* = R^* T^*$

Theorem 5.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $T \in B(\mathcal{H}, \mathcal{K})$.*

1. $(T^*)^* = T$.
2. $\| T^* \| = \| T \|$.
3. *The function $f : B(\mathcal{H}, \mathcal{K}) \to B(\mathcal{K}, \mathcal{H})$ defined by $f(R) = R^*$ is continuous.*
4. $\| T^*T \| = \| T \|^2$.

Lemma 5.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $T \in B(\mathcal{H}, \mathcal{K})$.*

1. $Ker(T) = (Im(T^*))^{\perp}$
2. $Ker(T^*) = (Im(T))^{\perp}$
3. $Ker(T^*) = 0$ *if and only if Im(T) is dense in $\mathcal{K}$.*

Lemma 5.6. *If $\mathcal{H}$ is a complex Hilbert space and $T \in B(\mathcal{H})$ is invertible then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.*

Theorem 5.7. *Let $\mathcal{H}$ be a complex Hilbert space.*

1. *If $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$ there is an orthogonal projection $P_{\mathcal{M}} \in B(\mathcal{H})$ with range $\mathcal{M}$, kernel $\mathcal{M}^{\perp}$ and $\| P_{\mathcal{M}} \| \leq 1$.*
2. *If Q is an orthogonal projection in $B(\mathcal{H})$ then Im(Q) is a closed linear subspace and $Q = P_{Im(Q)}$.*

Corollary 5.8. *If $\mathcal{H}$ is a complex Hilbert space, $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$, $\{e_n\}_{n=1}^{J}$ is an orthonormal basis for $\mathcal{M}$, where J is a positive integer or ∞, and P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, then*

$$Px = \sum_{n=1}^{J} (x, e_n) e_n.$$

6 Compact Operators

Definition 6.1. Let X and Y be normed spaces. A linear transformation $T \in L(X, Y)$ is *compact* if, for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ in Y contains a convergent subsequence. The set of compact transformations in $L(X, Y)$ will be denoted by $K(X, Y)$.

Theorem 6.2. *Let X and Y be normed spaces an let $T \in K(X, Y)$. Then T is bounded. Thus $K(X, Y) \subset B(X, Y)$.*

Proof. Suppose that T is not bounded. Then for each integer $n \geq 1$ there exists a unit vector x_n such that $\| Tx_n \| \geq n$. Since the sequence $\{x_n\}$ is bounded, by the compactness of T there exists a subsequence $\{Tx_{n(r)}\}$ which converges. This contradicts $\| Tx_{n(r)} \| \geq n(r)$. (ie. convergence implies boundedness) □

Theorem 6.3. *Let X, Y, Z be normed spaces*

1. *If $S, T \in K(X, Y)$ and $\alpha, \beta \in \mathbb{C}$ then $\alpha S + \beta T$ is compact. Thus K(X,Y) is a linear subspace of B(X,Y).*

2. *If $S \in B(X, Y)$, $T \in B(Y, Z)$ and at least on of the operators S, T is compact, then $TS \in B(X, Z)$ is compact.*

Proof.

1. Let $\{x_n\}$ be a bounded sequence in X. Since S is compact, there is a subsequence $\{x_{n(r)}\}$ such that $\{Sx_{n(r)}\}$ converges. Then, since $\{x_{n(r)}\}$ is bounded and T is compact, there is a subsequence $\{x_{n(r(s))}\}$ of the sequence $\{x_{n(r)}\}$ such that $\{Tx_{n(r(s))}\}$ converges.Since the sum of convergent sequences converges, it follows that the sequence $\{\alpha Sx_{n(r(s))} + \beta Tx_{n(r(s))}\}$ converges. Thus $\alpha S + \beta T$ is compact.

2. Let $\{x_n\}$ be a bounded sequence in X. If S is compact then there is a subsequence $\{x_{n(r)}\}$ such that $\{Sx_{n(r)}\}$ converges. Since T is bounded (and so is continuous), the sequence $\{TSx_{n(r)}\}$ converges. Thus TS is compact. If S is bounded but not compact the the sequence $\{Sx_n\}$ is bounded. Then since T must be compact, there is a subsequence $\{Sx_{n(r)}\}$ such that $\{TSx_{n(r)}\}$ converges, and again TS is compact.

□

For simplicity of notation we will make the following change (when it does not obscure the meaning of statements) $\{x_{n(r)}\}$, $\{x_{n(r(s))}\} \longrightarrow \{x_n\}$.

Theorem 6.4. *Let X, Y be normed spaces and $T \in B(X, Y)$.*

1. *If T has finite rank then T is compact.*

2. *If either dim(X) or dim(Y) is finite then T is compact.*

Proof.

1. Since T has finite rank, the space $Z = \operatorname{Im} T$ is a finite-dimensional normed space. Furthermore, for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ is bounded in Z, so by the Bolzano-Weierstrass theorem (1.10) this sequence must contain a convergent subsequence. Hence T is compact.

2. If dim X is finite then $r(T) \leq$ dim X, so $r(T)$ is finite, while if dim Y is finite then clearly the dimension of Im $T \subset Y$ must be finite. Thus, in either case the result follows from the previous part of this proof.

□

Theorem 6.5. *If X is an infinite-dimensional normed space then the identity operator I on X is not compact.*

Proof. Since X is an infinite-dimensional normed space the proof of Theorem 2.9 (see Thm 2.26 in [1] for details) shows there exists a sequence of unit vectors $\{x_n\}$ in X which does not have any convergent subsequence. Hence the sequence $\{Ix_n\} = \{x_n\}$ cannot have a convergent subsequence, and so the operator I is not compact. □

Corollary 6.6. *If X is an infinite-dimensional normed space and $T \in K(X)$ then T is not invertible.*

Proof. Suppose that T is invertible. Then, by Theorem 6.3, the identity operator $I = T^{-1}T$ on X must be compact. Since X is infinite-dimensional this contradicts Theorem 6.5. □

Theorem 6.7. *Let X, Y be normed spaces and let $T \in L(X, Y)$.*

1. *T is compact if and only if, for every bounded subset $A \subset X$, the set $T(A) \subset Y$ is relatively compact.*

2. *If T is compact the Im(T) and $\overline{Im(T)}$ are separable.*

Proof. 1. Suppose that T is compact. Let $A \subset X$ be bounded and suppose that $\{y_n\}$ is an arbitrary sequence in $\overline{T(A)}$. Then for each $n \in \mathbb{N}$, there exists $x_n \in A$ such that $\| y_n - Tx_n \| < n^{-1}$, and the sequence $\{x_n\}$ is bounded since A is bounded. Thus, by compactness of T, the sequence $\{Tx_n\}$ contains a convergent subsequence, and hence $\{y_n\}$ contains a convergent subsequence with limit in $\overline{T(A)}$. Since $\{y_n\}$ is arbitrary, this shows that $\overline{T(A)}$ is compact. Now suppose that for every bounded subset $A \subset X$ the set $T(A) \subset Y$ is relatively compact. Then for any bounded sequence $\{x_n\}$ in X the sequence $\{Tx_n\}$ lies in a compact set, and hence contains a convergent subsequence. Thus T is compact.

2. For any $r \in \mathbb{N}$, let $R_r = T(B_r(0)) \subset Y$ be the image of the ball $B_r(0) \subset X$. Since T is compact, the set R_r is relatively compact and so is separable, by Theorem 1.14. Furthermore, since Im T equals the countable union $\bigcup_{r=1}^{\infty} R_r$, it must also be separable. Finally, if a subset of Im T is dense in Im T then it is also dense in $\overline{\text{Im } T}$ (ie. by Definition 1.7.6 given $S \subset$ Im T we have $\overline{S} =$ Im T, and Im $T \subset \overline{\text{Im } T}$), so $\overline{\text{Im } T}$ is separable.

□

Theorem 6.8. *If X is a normed space, Y is a Banach space and $\{T_k\}$ is a sequence in K(X,Y) which converges to an operator $T \in B(X, Y)$, then T is compact. Thus K(X,Y) is closed in B(X,Y).*

Proof. Let $\{x_n\}$ be a bounded sequence in X. By compactness, there exists a subsequence of $\{x_n\}$, which we will label $x_{n(1,r)}$ (where $r = 1, 2, \ldots$), such that the sequence $\{T_1 x_{n(1,r)}\}$ converges. Similarly, there exists a subsequence $\{x_{n(2,r)}\}$ of $\{x_{n(1,r)}\}$ such that $\{T_2 x_{n(2,r)}\}$ converges. Also, $\{T_1 x_{n(2,r)}\}$ converges since it is a subsequence of $\{T_1 x_{n(1,r)}\}$. Repeating this process inductively, we see that for each $j \in \mathbb{N}$ there is a subsequence $\{x_{n(j,r)}\}$ with the property: for any $k \leq j$ the sequence $\{T_k x_{n(j,r)}\}$ converges. Letting $n(r) = n(r,r)$, for $r \in \mathbb{N}$, we obtain a single sequence $\{x_{n(r)}\}$ with the property that, for each fixed $k \in \mathbb{N}$, the sequence $\{T_k x_{n(r)}\}$ converges as $r \longrightarrow \infty$. This so-called "Cantor diagonalization" type argument is necessary to obtain a single sequence which works simultaneously for all the operators T_k, $k \in \mathbb{N}$.

We will now show that the sequence $\{T x_{n(r)}\}$ converges. We do this by showing that $\{T x_{n(r)}\}$ is a Cauchy sequence, and hence is convergent since Y is a Banach space.

Let $\epsilon > 0$ be given. Since the subsequence $\{x_{n(r)}\}$ is bounded there exists $M > 0$ such that $\| x_{n(r)} \| \leq M$, for all $r \in \mathbb{N}$. Also, since $\| T_k - T \| \longrightarrow 0$, as $k \longrightarrow \infty$, there exists an integer $K \geq 1$ such that $\| T_K - T \| < \frac{\epsilon}{3M}$. Next, since $\{T_K x_{n(r)}\}$ converges there exists an integer $R \geq 1$ such that if $r, s \geq R$ then $\| T_K x_{n(r)} - T_K x_{n(s)} \| < \frac{\epsilon}{3}$. Now we have, for r, $s \geq R$

$$
\begin{aligned}
&\| T x_{n(r)} - T x_{n(s)} \| \\
&< \| T x_{n(r)} - T_K x_{n(r)} \| + \| T_K x_{n(r)} - T_K x_{n(s)} \| + \| T_K x_{n(s)} - T x_{n(s)} \| \\
&\leq \| T_K - T \| \| x_{n(r)} \| + \frac{\epsilon}{3} + \| T_K - T \| \| x_{n(s)} \| \\
&< \frac{\epsilon}{3M} M + \frac{\epsilon}{3} + \frac{\epsilon}{3M} M \\
&= \epsilon
\end{aligned}
$$

which proves that $\{T x_{n(r)}\}$ is a Cauchy sequence. □

Corollary 6.9. *If X is a normed space, Y is a Banach space and $\{T_k\}$ is a sequence of bounded, finite rank operators which converge to $T \in B(X,Y)$, then T is compact.*

Proof. Theorem 6.4.2 shows $\{T_k\}$ are compact then we apply Theorem 6.8 □

Theorem 6.10. *If X is a normed space, $\mathcal{H}$ is a Hilbert space and $T \in K(X, \mathcal{H})$, then there is a sequence of finite rank operators $\{T_k\}$ which converges to T in $B(X,\mathcal{H})$.*

Proof. If T itself had finite rank the result would be trivial, so we consider the case that it does not. By Lemma 3.6 and Theorem 6.7 the set $\overline{\text{Im}\, T}$ is an infinite-dimensional, separable Hilbert space, so by Theorem 3.16 it has an orthonormal basis $\{e_n\}$. For each integer $k \geq 1$, let P_k be the orthogonal projection from $\overline{\text{Im}\, T}$ onto the linear subspace $\mathcal{M}_k = \text{Sp}\{e_1, \ldots, e_k\}$, and let $T_k = P_k T$. Since $\text{Im}\, T_k \subset \mathcal{M}_k$, the operator T_k has finite rank. We will show that $\| T_k - T \| \longrightarrow 0$ as $k \longrightarrow \infty$.

Suppose that this is not true. Then, after taking a subsequence of the sequence $\{T_k\}$ if necessary, there is an $\epsilon > 0$ such that $\| T_k - T \| \geq \epsilon$ for all k. Thus there exists a sequence of unit vectors $x_k \in X$ such that $\| (T_k - T) x_k \| \geq \frac{\epsilon}{2}$ for all k. Since T is compact, we may suppose that $T x_k \longrightarrow y$, for some $y \in \mathcal{H}$

(after again taking a subsequence, if necessary). Now, using the representation of P_m in Corollary 5.8, we have,

$$\begin{aligned}(T_k - T)x_k &= (P_k - I)Tx_k \\ &= (P_k - I)y + (P_k - I)(Tx_k - y) \\ &= -\sum_{n=k+1}^{\infty} (y, e_n)e_n + (P_k - I)(Tx_k - y).\end{aligned}$$

Hence, by taking the norms and using $\| P_k \|= 1$ (from Theorem 5.7) we deduce (using properties of norms) that

$$\frac{\epsilon}{2} \leq \| (T_k - T)x_k \| \leq (\sum_{n=k+1}^{\infty} (y, e_n)^2)^{\frac{1}{2}} + 2 \| Tx_k - y \| .$$

The right-hand side of this inequality tends to zero as $k \longrightarrow \infty$, which is a contradiction, and so proves the theorem. □

Lemma 6.11. *If $\mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$, then $r(T) = r(T^*)$ (either as finite numbers or as ∞). In particular, T has finite rank if and only if T^* has finite rank.*

Proof. Suppose first that $r(T) < \infty$. For any $x \in \mathcal{H}$, we write that the orthogonal decomposition of x with respect to Ker T^* as $x = u + v$, with $u \in$ Ker T^* and $v \in (\text{Ker}T^*)^{\perp} = \overline{\text{Im}T} = \text{Im}T$ (since $r(T) < \infty$). Thus $T^*x = T^*(u + v) = T^*v$, and hence $\text{Im}T^* = T^*(\text{Im}T)$, which implies that $r(T^*) \leq r(T)$. Thus, $r(T^*) \leq r(T)$ when $r(T) < \infty$.

Applying this result to T^*, and using $(T^*)^* = T$, we also see that $r(T) \leq r(T^*)$ when $r(T^*) < \infty$. This proves the lemma when both the ranks are finite, and also shows that it is impossible for one rank to be finite and the other infinite, and so also proves the infinite rank case. □

Theorem 6.12. *If $\mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$, then T is compact if and only if T^* is compact.*

Proof. Suppose that T is compact. Then by Theorem 6.10 there is a sequence of finite rank operators $\{T_n\}$, such that $\| T_n - T \| \longrightarrow 0$. By Lemma 6.11, each operator T_n^* has finite rank and, by Theorem 5.4, $\| T_n^* - T^* \| = \| T_n - T \| \longrightarrow 0$. Hence it follows from Corollary 6.9 that T^* is compact. Thus, if T is compact then T^* is compact. It now follows from this result and $(T^*)^* = T$ that if T^* is compact the T is compact, which completes the proof. □

Constructing a Compact Operator from Finite Rank Operators

One can build the Compact Operator $T \in B(\ell^2)$ defined by $T\{a_n\}\{n^{-1}a_n\}$ from the finite rank operators $T_k \in B(\ell^2)$ where $T_k\{a_n\} = \{b_n^k\}$ and $b_n^k = n^{-1}a_n$ when $n \leq k$ and $b_n^k = 0$ when $n > k$. Thus for any $a \in \ell^2$

$$\| (T_k - T)a \|^2 = \sum_{n=k+1}^{\infty} \frac{|a_n|^2}{n^2} \leq (k+1)^{-2} \sum_{n=k+1}^{\infty} |a_n|^2 \leq \frac{\| a \|^2}{(k+1)^{-2}}$$

and therefore

$\| T_k - T \| \leq (k+1)^{-1} \longrightarrow 0$

The result follows from Corollary 6.9.

7 Spectral Theory of Compact Operators

Definition 7.1. Let $\mathcal{H}$ be a Hilbert space and let $S \in B(\mathcal{H})$. We define the sets

$$\begin{aligned}
\sigma_(S) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\} \\
\sigma_p(S) &= \{\lambda : \lambda \text{ is an eigenvalue of S}\} \\
\rho(S) &= \mathbb{C} \backslash \sigma(S)
\end{aligned}$$

We denote the *spectrum* of T by $\sigma(T)$, the set $\sigma_p(S)$ is the *point spectrum* of S, and $\rho(S)$ is the *resolvent set* of S.

Theorem 7.2. *If $\mathcal{H}$ is infinite-dimensional then $0 \in \sigma(T)$. If $\mathcal{H}$ is separable then either $0 \in \sigma_p(T)$ or $0 \in \sigma(T) \backslash \sigma_p(t)$ may occur. If $\mathcal{H}$ is not separable then $0 \in \sigma_p(T)$.*

Proof. See [1] □

Theorem 7.3. *If $\lambda \neq 0$ then Ker(T - λI) has finite dimension.*

Proof. Suppose that $\mathcal{M} = \mathrm{Ker}(T - \lambda I)$ is infinite-dimensional. Since the kernel of a bounded operator is closed (by Lemma 4.6), the space $\mathcal{M}$ is an infinite-dimensional Hilbert space, and there is an orthonormal sequence $\{e_n\}$ in $\mathcal{M}$ (by Theorem 3.15). Since $e_n \in \mathrm{Ker}(T - \lambda I)$ we have $Te_n = \lambda e_n$ for each $n \in \mathbb{N}$, and since $\lambda \neq 0$ the sequence $\{\lambda e_n\}$ cannot have a convergent subsequence, since $\{e_n\}$ is orthonormal (a contradiction arises as the subsequence would be Cauchy but for any 2 elements $\| e_n - e_m \| = \sqrt{2}$). This contradicts the compactness of T, which proves the theorem. □

Theorem 7.4. *If $\lambda \neq 0$ then Im(T - λI) is closed.*

Proof. Let $\{y_n\}$ be a sequence in $\mathrm{Im}(T - \lambda I)$, with $\lim_{n \to \infty} y_n = y$. Then for each n we have $y_n = (T - \lambda I)x_n$, for some x_n, and since $\mathrm{Ker}(T - \lambda I)$ is closed, x_n has an orthoganal decomposition of the form $x_n = u_n + v_n$, with $u_n \in \mathrm{Ker}(T - \lambda)$ and $v_n \in \mathrm{Ker}(T - \lambda I)^{\perp}$. We will show that the sequence $\{v_n\}$ is bounded.

Suppose not. Then, after taking a subsequence if necessary, we may suppose that $\| v_n \| \neq 0$, for all n, and $\lim_{n \to \infty} \| v_n \| = \infty$. Putting $w_n = \frac{v_n}{\| v_n \|}$, $n = 1, 2, \ldots$, we have $w_n \in \mathrm{Ker}(T - \lambda I)^{\perp}$, $\| w_n \| = 1$ (so the sequence $\{w_n\}$ is bounded) and

$$(T - \lambda I)w_n = \frac{y_n}{\| v_n \|} \longrightarrow 0,$$

since $\{y_n\}$ is bounded (because it is convergent). Also, by the compactness of T we may suppose that $\{Tw_n\}$ converges (after taking a subsequence if necessary). By combining these results it follows that the sequence $\{w_n\}$ converges (since $\lambda \neq 0$). Letting $w = \lim_{n \to \infty} w_n$, we see that $\| w \| = 1$ and

$$(T - \lambda I)w = \lim_{n \to \infty} (T - \lambda I)w_n = 0,$$

so $w \in \mathrm{Ker}(T - \lambda I)$. However, $w_n \in \mathrm{Ker}(T - \lambda I)^{\perp}$ so

$$\| w - w_n \|^2 = (w - w_n, w - w_n) = 1 + 1 = 2,$$

which contradicts $w_n \longrightarrow w$. Hence the sequence $\{v_n\}$ is bounded.

Now, by the compactness of T we may suppose that $\{Tv_n\}$ converges. Then $v_n = \lambda^{-1}(Tv_n - (T-\lambda I)v_n) = \lambda^{-1}(Tv_n - y_n)$, for $n \in \mathbb{N}$, so the sequence $\{v_n\}$ converges. Let its limit be v. Then

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty}(T-\lambda I)v_n = (T-\lambda I)v,$$

and so $y \in \mathrm{Im}(T-\lambda I)$. This proves that $\mathrm{Im}(T-\lambda I)$ is closed. □

Corollary 7.5. *If $\lambda \neq 0$ then*

$$Im(T-\lambda I) = Ker(T^* - \bar{\lambda} I)^{\perp}, \qquad Im(T^* - \bar{\lambda} I) = Ker(T-\lambda)^{\perp}$$

Proof. Since T^* is compact we apply Theorems 7.3 and 7.4 showing $\mathrm{IM}(T^* - \overline{\lambda} I)$ is closed when $\lambda \neq 0$. Thus applying Corollary 3.12 and Lemma 5.5 we prove our result. □

Theorem 7.6. *For any real $t > 0$, the set of all distinct eigenvalues λ of T with $|\lambda| \geq t$ is finite.*

Proof. Suppose instead that for some $t_0 > 0$ there is a sequence of distinct eigenvalues $\{\lambda_n\}$ with $|\lambda_n| \geq t_0$ for all n, and let $\{e_n\}$ be a sequence of corresponding unit eigenvectors. We will now construct, inductively, a particular sequence of unit vectors$\{y_n\}$. Let $y_1 = e_1$. Now consider any integer $k \geq 1$. By Lemma 1.5 the set $\{e_1, \ldots, e_k\}$ is linear independent, thus the set $\mathcal{M}_k = \mathrm{Sp}\{e_1, \ldots, e_k\}$ is k-dimensional and so is closed by Corollary 2.8. Any $e \in \mathcal{M}_k$ can be written as $e = \alpha_1 e_1 + \ldots + \alpha_k e_k$, and we have

$$(T-\lambda_k I)e = \alpha_1(\lambda_1 - \lambda_k)e_1 + \ldots + \alpha_{k-1}(\lambda_{k-1} - \lambda_k)e_{k-1},$$

and so if $e \in \mathcal{M}_k$,

$$(T-\lambda_k I)e \in \mathcal{M}_{k-1}.$$

Similarly, if $e \in \mathcal{M}_k$,

$$Te \in \mathcal{M}_k.$$

Next, $\mathcal{M}_k$ is a closed subspace of $\mathcal{M}_{k+1}$, so the orthogonal complement of $\mathcal{M}_k$ in $\mathcal{M}_{k+1}$ is a non-trivial linear subspace of $\mathcal{M}_{k+1}$. Hence there is a unit vector $y_{k+1} \in \mathcal{M}_{k+1}$ such that $(y_{k+1}, e) = 0$ for all $e \in \mathcal{M}_k$, and $\| y_{k+1} - e \| \geq 1$. Repeating this process inductively, we construct a sequence $\{y_n\}$.

It now follows from the construction of the sequence $\{y_n\}$ that for any integers m, n with $n > m$,

$$\| Ty_n - Ty_m \| = |\lambda_n| \, \| y_n - \lambda_n^{-1}[-(T-\lambda_n)y_n + Ty_m] \| \geq |\lambda_n| \geq t_0$$

since by the above results, $-(T-\lambda_n)y_n + Ty_m \in \mathcal{M}_{n-1}$. This shows that the sequence $\{Ty_n\}$ cannot have a convergent subsequence. This contradicts the compactness of T, and so proves the theorem. □

Corollary 7.7. *The set $\sigma_p(T)$ is at most countably infinite. If $\{\lambda_n\}$ is any sequence of distinct eigenvalues of T then $\lim_{n\to\infty} \lambda_n = 0$.*

Proof. We apply the previous theorem to the union of finite sets of eigenvalues λ with $\lambda \geq r^{-1}$, with $r = 1, 2, \ldots$ $\square$

Lemma 7.8. *If T has finite rank and $\lambda \neq 0$, then either: (a) $\lambda \in \rho(T)$ and $\bar{\lambda} \in \rho(T^*)$; or (b) $\lambda \in \sigma_p(T)$ and $\bar{\lambda} \in \sigma_p(T^*)$. Furthermore,*

$$n(T - \lambda I) = n(T^* - \bar{\lambda} I) < \infty$$

Proof. Let $\mathcal{M} = \text{Im}T$ and $\mathcal{N} = \text{Ker}T^* = \mathcal{M}^\perp$ (by Lemma 5.5). Since $\mathcal{M}$ is finite-dimensional it is closed, co any $x \in \mathcal{H}$ has an orthogonal decomposition $x = u + v$, with $u \in \mathcal{M}$, $v \in \mathcal{N}$. Using this decomposition, we can identify any $x \in \mathcal{H}$ with a unique element $(u, v) \in \mathcal{M} \times \mathcal{N}$, and vice versa (alternatively, this shows that the space $\mathcal{H}$ is isometrically isomorphic to the space $\mathcal{M} \times \mathcal{N}$). Also,

$$(T - \lambda I)(u + v) = Tu - \lambda u + Tv - \lambda v,$$

and we have $Tu - \lambda u \in \mathcal{M}$, $Tv - \lambda v \in \mathcal{N}$. It follows from this that we can express the action of the operator $(T - \lambda I)$ in matrix form by

$$(T - \lambda I) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (T - \lambda I)|_{\mathcal{M}} & T|_{\mathcal{N}} \\ 0 & -\lambda I|_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where $(T - \lambda I)|_{\mathcal{M}} \in B(\mathcal{M})$, $T|_{\mathcal{N}} \in B(\mathcal{N}, \mathcal{M})$, and $I|_{\mathcal{N}} \in B(\mathcal{N})$ denote the restrictions of the operators $T - \lambda I$, T, and I to the spaces $\mathcal{M}$ and $\mathcal{N}$. We now write $A = (T - \lambda I)|_{\mathcal{M}}$. It follows from Lemma 1.4 and Corollary 4.11 that either A is invertible ($n(A) = 0$) or $n(A) > 0$, and so, either $T - \lambda I$ is invertible or $n(T - \lambda I) = n(A) > 0$ (see Exercise 7.18 of [1]), that is, either $\lambda \in \rho(T)$ or $\lambda \in \sigma_p(T)$.

Now let $P_{\mathcal{M}}$, $P_{\mathcal{N}}$ denote the orthogonal projections of $\mathcal{H}$ onto $\mathcal{M}$, $\mathcal{N}$. Using $I = P_{\mathcal{M}} + P_{\mathcal{N}}$ and $\mathcal{N} = \text{Ker}T^*$, we have

$$(T^* - \overline{\lambda} I)(u + v) = (T^* - \overline{\lambda} I)u - \overline{\lambda} v = P_{\mathcal{M}}(T^* - \overline{\lambda} I)u + P_{\mathcal{N}} T^* u - \overline{\lambda} v.$$

Hence $T^* - \overline{\lambda} I$ can be represented in matrix form by

$$(T^* - \overline{\lambda} I) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} P_{\mathcal{M}}(T^* - \overline{\lambda} I)|_{\mathcal{M}} & 0 \\ P_{\mathcal{N}}(T^*)|_{\mathcal{M}} & -\overline{\lambda} I|_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Also, $A^* = P_{\mathcal{M}}(T^* - \overline{\lambda} I)|_{\mathcal{M}} \in B(\mathcal{M})$ as, for all u, $w \in \mathcal{M}$, $(Au, w) = ((T - \lambda I)u, w) = (u, (T^* - \overline{\lambda} I)w) = (P_{\mathcal{M}} u, (T^* - \overline{\lambda} I)w) = (u, P_{\mathcal{M}}(T^* - \overline{\lambda} I)w)$. Again by finite-dimensional linear algebra, $n(A^*) = n(A)$. It now follows (see Exercise 7.18 of [1]) that if $n(A) = 0$ then $T - \lambda I$ and $T^* - \overline{\lambda} I$ are invertible, while if $n(A) > 0$ then $n(T - \lambda I) = n(T^* - \overline{\lambda} I) = n(A) > 0$, so $\lambda \in \sigma_p(T)$ and $\overline{\lambda} \in \sigma_p(T^*)$. $\square$

Lemma 7.9. *Let X be a normed space and S, $A \in B(X)$, with A invertible. Let $T = SA$. Then*

1. *T is invertible if and only if S is*
2. *$n(T = n(S)$*

Proof. 1. If S is invertible then $A^{-1}S^{-1}$ is a bounded inverse for T , so T is invertible. Also, $S = TA^{-1}$, so a similar argument shows that if T is invertible then S is invertible

2. $x \in KerT$ if and only if $Ax \in KerS$. A is invertible and thus full rank, so it preserves the dimension of the null-space. giving the result.

$\square$

Theorem 7.10. *If T is compact and $\lambda \neq 0$, then either (a) $\lambda \in \rho(T)$ and $\bar{\lambda} \in \rho(T^*)$; or (b) $\lambda \in \sigma_p(T)$ and $\bar{\lambda} \in \sigma_p(T^*)$. Furthermore,*

$$n(T - \lambda I) = n(T^* - \bar{\lambda} I) < \infty$$

Proof. We first reduce the problem to the case of a finite rank operator. By Theorem 6.10 there is a finite operator T_F on $\mathcal{H}$ with $\| \lambda^{-1}(T - T_F) \| < \frac{1}{2}$, so by Theorem 4.8 and Lemma 5.6, the operators $S = I - \lambda^{-1}(T - T_F)$ and S^* are invertible. Now, letting $G = T_F S^{-1}$ we see that

$$T - \lambda I = (G - \lambda I)S, \text{ and so } T^* - \overline{\lambda} I = S^*(G^* - \overline{\lambda} I).$$

Since S and S^* are invertible it follows that $T - \lambda I$ and $T^* - \overline{\lambda} I$ are invertible if and only if $G - \lambda I$ and $G^* - \overline{\lambda} I$ are invertible, and $n(T - \lambda I) = n(G - \lambda I)$, $n(T^* - \overline{\lambda} I) = n(G^* - \overline{\lambda} I)$ by Lemma 7.9. Now, since $\text{Im}G \subset \text{Im}T_F$ the operator G has finite rank, so the first results of the theorem follow from Lemma 7.8. $\square$

This can be restated as follows.

Theorem 7.11. *Given $\lambda \in \mathbb{C}$, $\mathcal{H}$ is an infinite dimensional Hilbert space, $T \in K(\mathcal{H})$*

- *If $\lambda = 0$ then $\lambda = 0 \in \sigma(T)$*
- *If $\lambda \neq 0$, $\lambda \in \sigma(T)$ then $\lambda \in \sigma_p(T)$ (ie is an eigenvalue).*

which together implies,

- *$\sigma(T) = \{0\} \bigcup \sigma_p(T)$*
- *ie a non zero complex number in the spectrum of a compact operator is and eigenvalue*

Consider the following equations:

$$(T - \lambda I)x = 0, \qquad (T^* - \bar{\lambda} I)y = 0, \tag{7.0.1a}$$
$$(T - \lambda I)x = p, \qquad (T^* - \bar{\lambda} I)y = q, \tag{7.0.1b}$$

Theorem 7.12. ***(The Fredholm Alternative)***
If $\lambda \neq 0$ then one or the other of the following alternatives holds.

1. *Each of the homogeneous equations* (7.0.1a) *has only the solution $x = 0$, $y = 0$, respectively, while the corresponding inhomogeneous equations* (7.0.1b) *have unique solutions x, y for any given $p, q \in \mathcal{H}$.*

2. *There is a finite number $m_\lambda > 0$ such that each of the homogeneous equations* (7.0.1a) *has exactly m_λ linearly independent solutions, say x_n, y_n, $n = 1, \ldots, m_\lambda$, respectively, while the corresponding inhomogeneous equations* (7.0.1b) *have solutions if and only if p, $q \in \mathcal{H}$ satisfy the conditions*

$$(p, y_n) = 0, \qquad (q, x_n) = 0, \qquad n = 1, \ldots, m_\lambda.$$

Proof. The result follows immediately from Theorem 7.10. Alternative (a) corresponds to the case $\lambda \in \rho(T)$, while alternative (b) corresponds to the case $\lambda \in \sigma_p(T)$. In this case, $m_\lambda = n(T - \lambda I)$. It follows from Corollary 7.5 that the conditrions on p, q in (b) ensure that $p \in \operatorname{Im}(T - \lambda I)$, $q \in \operatorname{Im}(T^* - \overline{\lambda} I)$, respectively, so solutions of (7.0.1b) exist. □

In applied math and physics many problems can be brought to the form

$$Ru = f \tag{7.0.2}$$

Where R is a linear operator and f is a give function (or data).

Definition 7.13. Equation (7.0.2) (or the corresponding physical model) is said to be *well-posed* if the following properties hold:

1. A solution u exists for every f.
2. The solution u is unique for each f.
3. The solution u depends continuously on f in a suitable sense.

An Operator with no Eigenvalues but which does have a Spectrum

The Linear Operator $S : \ell^2 \longrightarrow \ell^2$ defined by $S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$ is called the *unilateral shift operator*. Its norm is given by $\| S \| = Sup\{|S(x)| : \| x \| \leq 1, x \in \ell^2\}$.

Result

The unilateral shift has no eigenvalues ($\sigma_p = \varnothing$).

Proof. Consider λ and eigenvalue of S with corresponding non-zero eigenvector $x = \{x_n\}$. Then

$$S(x) = (0, x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$$

If $\lambda = 0$ then x must be the zero vector, and if $\lambda \neq 0$ x must still be the zero vector. x is non-zero so no eigenvalues are valid. □

The unilateral shift does have a spectrum though (See Example 6.38 in [1]). In fact $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

Remark

It would be nice to have a class of operators which was guaranteed to have $\sigma(T) = \sigma_p(T)$ or at least something close to it. A good choice is the Compact Operators. They have $\sigma(T) = \{0\} \bigcup \sigma_p(T)$.

8 Applications

The previous results can be used to solve a variety of problems in applied math and physics, Fredholm and Volterra Integral Equations, Differential Equations, Eigenvalue Problems and Green's Functions, Dirichlet problem for Heat, Wave, and Laplace Equations to name a few. These would make interesting future avenues of research.

Furthermore the C^*-Algebra of Compact Operators is isomorphic to the Density Operators from Quantum Mechanics. These operators are used to describe impure (entangled states) and are commonly part of graduate courses in Quantum Mechanics. It would be interesting to explore the connections between the fields at some point.

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