Branching Processes and Applications in Macroeconomy

by
Zenghui Li

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Lakehead University
Thunder Bay, Ontario, Canada

Abstract

My honor project is on branching process, which is a special case of Markov process. The branching process is a stochastic counting process that arises naturally in a large variety of daily-life situations. My main focus is to understand branching process. In this project, a brief historical review of this topic is provided, some basic concepts, tools, several properties, and theoretical approaches for studying Branching processes are introduced/discussed. We further briefly discuss how to use branching process to model the effect of fiscal multiplier in macroeconomy.

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CHAPTER 1

History of Branching Processes and Preliminaries

The purpose of this project will be introducing branching process, and making an application of the properties covered. Branching process is also useful to model species extinction, infectious diseases propagation, and many other phenomena. We will show that we can use branching process to model the fiscal multiplier and the total consuming impact on the economy. The object of this chapter is to introduce the history of branching processes and some preliminaries to help understand the following contents.

1. Branching Process

This section is devoted to help understand some historic review of stochastic processes, Markov chain and branching processes.

1.1. Stochastic Processes.

Stochastic process problems are concerning about the models for systems that evolve unpredictably in time. This type of models is essential in diverse fields such as economics, finance, physics, climatology, telecommunication, biology, etc. The following content will give some general concepts to these kind of models. The main function of stochastic process study is to formulate practical models into probability statements and use the knowledge of probability to solve them. Moreover, as the concepts and techniques developed under this topic are also found in other areas such as Time Series Analysis, Risk Theory, etc.

The topic of stochastic process was studied in the analysis of physics and statistical mechanics at the very beginning. Around 1907, A.Andreyevich Markov gave his research on stochastic model, which was called *Markov Chains* and *Markov Processes*. The first announcement of stochastic process could be tracked back to 1930s, two mathematicians found out the similar theory almost at the same time. The Russian mathematician A.N.Kolmogorov and U.S. mathematician N.Wiener introduced the concepts of stationary processes, and from then, the topic of stochastic process was well developed.

1.2. Markov Processes.

We will only focus on *discrete-time Markov Chain* in this section.

DEFINITION 1.1. A discrete-time Markov chain is a sequence of random variables X_1, X_2, X_3, ... with the Markov property, namely that the probability of moving to the next state depends only on the present state and not on the previous states

$$Pr(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, ..., X_1 = x_1) = Pr(X_{n+1} = x_{n+1} | X_n = x_n)$$

for all X_{n+1}, X_n,...,$X_1 \in S$ and n≥1.
Where the possible values of X_i form a countable set S called the state space of the chain.

1.3. Branching Processes.

In probability theory, a branching process is a *Markov process* that models a population in which each individual in generation n produces some random number of individuals in generation $n + 1$, according, in the simplest case, to a fixed probability distribution that does not vary from individual to individual. Branching processes are used to model reproduction; for example, the individuals might correspond to bacteria, each of which generates 0, 1, or 2 offspring with some probability in a single time unit. Branching processes can also be used to model other systems with similar dynamics, e.g., the spread of surnames in genealogy or the propagation of neutrons in a nuclear reactor.

The study of branching processes began in the 1840s with J.Bienaym, a probabilist and statistician, and was advanced in the 1870s with the work of Reverend.H.W.Watson, a clergyman and mathematician, and Francis Galton, a biometrician. In 1873, Galton sent a problem to the Educational Times regarding the survival of family names.When he did not receive a satisfactory answer, Galton consulted Watson, who rephrased the problem in terms of generating functions. The simplest and most frequently applied branching processes models were named after Galton and Watson, a type of discrete-time Markov chains. Branching processes fit under the general heading of stochastic processes.

2. Preliminaries

Before Studying the properties of branching processes, we need to review some properties of probability distributions, and give some examples that follow different kind of probability distributions. In probability and statistics, a probability distribution is a mathematical function that, stated in simple terms, can be thought of as providing the probability of occurrence of different possible outcomes in an experiment. If the random variable X is used to denote the outcome of a balanced coin toss ("the experiment"), then the probability distribution of X takes the value 0.5 for X=Heads, and 0.5 for $X = Tails$.

DEFINITION 1.2. A *sample space* of an random experiment is a set that includes all possible outcomes.

DEFINITION 1.3. Let Ω be the sample space of an experiment, $\Omega = \omega$. If for every $\omega \in \Omega$, there exists a real number $X(\omega)$ relate to it, then denote $X = X(\omega)$, $\omega \in \Omega$ is a *random variable*.

2.1. Continuous and Discrete Probability Distributions.

To fully explain the random variable, we shall describe the probability of its' each possible value.

DEFINITION 1.4. If possible values of the random variable X is finite or infinite countable possible values then we say X is a *discrete random variable.*

DEFINITION 1.5. Say all possible values of a discrete random variable X are x_k (k=1,2,3....), then the probability of event $X = x_k$ is given by

$$p_k = P(X = x_k), k = 1, 2, 3...$$

and p_k has the following two properties:

$$p_k \geq 0, k = 1, 2, 3....$$

and

$$\sum_{k=1}^{\infty} p_k = 1$$

then we say $p_k = P(X = x_k), k = 1, 2, 3...$ is the probability distribution of discrete random variable X.

EXAMPLE 1.6. If a random variable X has probability distribution

$$P(X = x_k) = \frac{1}{n}, k = 1, 2, 3, ..., n$$

, then we say X is discrete uniformly distributed.

DEFINITION 1.7. The cumulative distribution function of random variable X is defined by $F(X) = P(X \leq x)$, $-\infty < x < +\infty$, if there exist a non-negative integrable function $f(x)$ such that for $-\infty < x < +\infty$:

$$F(x) = \int_{-\infty}^{x} f(t)dt,$$

then X is a *continuous random variable*, and $f(x)$ is the *probability density function*, denoted by $X \sim f(x)$.
The function $f(x)$ will have the following properties:

$$f(x) \geq 0$$

$$F(+\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1$$

and for any real number a,b (a $<$ b) we have

$$P(a < X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx.$$

REMARK 1.8. If X is a continuous random variable, then for any real number a, we have $P(X = a) = 0$.

PROOF. For any $\Delta x > 0$ we have

$$0 \leq P(X = a) \leq P(a - \Delta x < X \leq a)$$
$$= \int_{a-\Delta x}^{a} f(x)dx \to 0(\Delta x \to 0),$$

that is

$$P(X = a) = 0.$$

□

EXAMPLE 1.9. Denote a passenger's waiting time (in minutes) for a bus as X, and it follows uniform distribution in (0,10), find the probability of passenger's waiting time is less than 5 minutes.

SOLUTION 1. The probability density function of X is:

$$f(x) = \frac{1}{10}, 0 < x < 10$$

So the probability of waiting time less than 5 minuets is

$$P(X \leq 5) = \int_{-\infty}^{5} f(x)dx = \int_{0}^{5} \frac{1}{10}dx = 0.5.$$

2.2. Binomial Distribution and Poisson Distribution.

The binomial distribution is frequently used to model the number of successes in a sample of size n drawn with replacement from a population of size N. If the sampling is carried out without replacement, the draws are not independent and the resulting distribution is a hyper-geometric distribution. However, for N much larger than n, the binomial distribution remains a good approximation.

DEFINITION 1.10. In probability theory and statistics, the binomial distribution with parameters n and p is the discrete probability distribution of the number of successes in a sequence of n independent yes/no experiments, each of which yields success with probability p. A success/failure experiment is also called a Bernoulli experiment or Bernoulli trial; when $n = 1$, the binomial distribution is a Bernoulli distribution. The binomial distribution is the basis for the popular binomial test of statistical significance.

DEFINITION 1.11. if the random variable X follows the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$, we write $X\ B(n, p)$. The probability of getting exactly k successes in n trials is given by the probability mass function:

$$f(k; n, p) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

for $k = 0.1, 2, 3, ..., n$

EXAMPLE 1.12. There are 10 machines in a factory. In the given period, the probability of requires repairing of each machine is 0.3, find the probability of at least one machine requires repair in the given period.

SOLUTION 2. Let x be the number of machines require repairing in the duration,

$$P(k \geq 1) = 1 - 0.7^{10} \approx 0.9718$$

Poisson distribution is a typical discrete distribution, recall the definition as follows.

DEFINITION 1.13. Suppose we conduct a Poisson experiment, in which λ is the average number of successes within a given region. Let X be the number of successes that result from the experiment. Then

$$f(x) = P(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}\ x = 0, 1, 2, ..., \ , \ \lambda > 0$$

where x is the number of successes and e is the natural logarithm. Then X has *Poisson distribution*, denoted by $X \sim \mathrm{P}_\lambda$.

EXAMPLE 1.14. From the past record, a shopping mall use Poisson distribution to describe the sales of a specific good with $\lambda = 10$, if the shopping mall will only replenish the goods in the end of every month, how many goods should the mall keep at the beginning of the month to have 95% confidence that the good will not be sold out?

SOLUTION 3. We assume the sales of this good per month is X, and the number of goods left in the end of month is a, then $X \leq a$ means that the good will not be sold out. We need to find the a such that

$$P(X \leq a) \geq 0.95.$$

,As we know that X follows the Poisson distribution with parameter $\lambda = 10$, we have that

$$\sum_{k=0}^{a} \frac{10^k}{k!}e^{10} \, .$$

From the Poisson distribution table, we have

$$\sum_{k=0}^{14} \frac{10^k}{k!}e^{10} \ \approx 0.9166 < 0.95.$$

and

$$\sum_{k=0}^{15} \frac{10^k}{k!}e^{10} \ \approx 0.9513 > 0.95.$$

So the mall has to keep no less than 15 goods to have 95% that the good will not be sold out.

CHAPTER 2

Galton-Watson Branching Processes

As the introduced history in Chapter 1, branching process was first announced by a biometrician, so this math technique was also first used in the study of bio-specious. In other words, at the beginning of branching processes study, the main topics are about the status and trending of a specious' population with some given information like the individuals' spawning ability and several assumptions.

1. Galton-Watson Branching Processes

1.1. Moment Generating Function of Branching Processes.

One of the fundamental tools required for studying branching processes is generating functions. Let X be a discrete random variable taking values in the set $\{0, 1, 2, 3, ...\}$ with associated probabilities given by

$$p_j = \text{Pr(X = j), j = 0, 1, 2, 3, ...}$$

(1.1)

Then the moment generation function and probability generation function of random variable X are defined, respectively, by

$m_X(\text{t}) = \text{E}(\ e^{tX}\ ) = \sum_{j=0}^{\infty} p_j\ e^{jt}$ and $\phi(\text{s}) = \text{E}(s^X\ ) = \sum_{j=0}^{\infty} p_j s^j.$

Clearly,

$$m_x(t) = \phi(e^t)$$

1.2. The Description of Branching Processes.

The most common formulation of a branching process is that of the Galton-Watson process.

DEFINITION 2.1. Let Z_n be the population size of a specious at time n or generation n, and let $X_{n,i}$ be a random variable denoting the number of direct successors of member i at time n or generation n, where $X_{n,i}$ are independent and identically distributed random variables over all $n \in \{0, 1, 2, 3, ...\}$ *and* $1 \leqslant$ i $\leqslant Z_n$.

Let

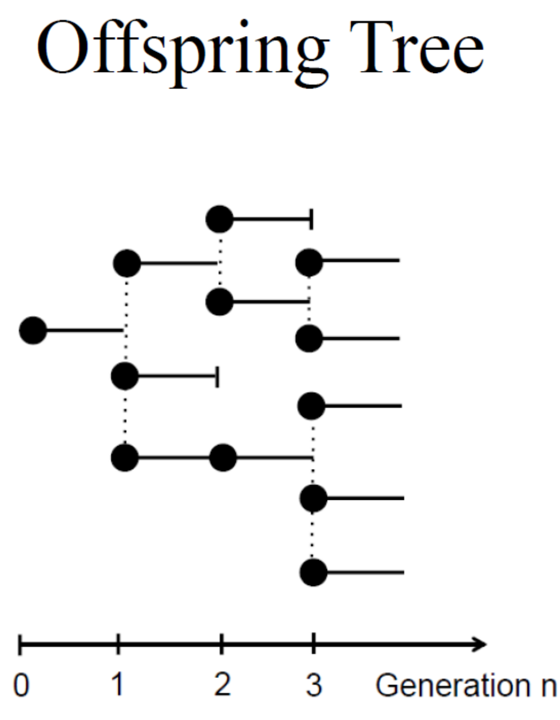
$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \text{ with } Z_0 = 1 \text{ and } Z_1 = X_{0,1}, n = 1, 2, ...$$

Then $\{ Z_n$: n $\geqslant 0\}$ is called a Galton-Watson branching process.

To understand the definition above better, we then explain the definition in different words as follows. In the description above, $Z_0 = 1$ indicates that there is only 1 individual of the *zero* generation, Z_1 is the total number of individuals in first generation, and $X_{0,1}$ indicates the descendants number of the 1st individual of *zero* generation.

1.3. Offspring Tree.

To demonstrate the branching process in a more direct way, we introduce the concept of Offspring Tree.



EXAMPLE 2.2. According to the offspring tree above, we can tell that there is only one individual in generation zero, that is Z_0=1. Form the figure above, we can see that there are 3 descendants from the very first individual, so we have Z_1=$X_{0,1}$=3. Similarly, the second individual of generation 2 has 2 descendants, so $X_{2,2}$=2.

REMARK 2.3. if an individual doesn't have any kids, the notation in offspring tree was shown as the 2nd individual of generation 1.

Before we can define the Galton Watson branching processes in next chapter, there are several important assumptions:

(1) Every individual lives exactly one unit of time, then produces X offspring and dies.
(2) The number of offspring, X, takes values 0,1,2,..., and the probability of producing k offspring is P(X=k) $=p_k$
(3) All individuals reproduce independently. Individuals 1,2,...,n have family sizes X_1, X_2,..., X_n, where each X_i has the same distribution as X.
(4) let Z_n be the number of individuals born at time n, for n = 0,1,2,... Interpret Z_n as the size of generation n.

DEFINITION 2.4. With all the assumptions, properties and notations above, we define the branching processes as $\{ Z_0, Z_1, Z_2, ... \} = \{ Z_n : n=0,1,2,... \}$

CHAPTER 3

Basic Properties of Galton-Watson Branching Processes

We will provide some basic properties of Galton-Watson branching processes in this chapter. Especially, we will discuss how to find the probability of extinction of any given Galton-Watson branching process.

1. Mathematically Define the Branching Process

1.1. Three Different Cases.

In this chapter we will study the basic properties and limit theorems of Galton-Watson branching process { Z_n: n $\geqslant 0$}.A central question in the theory of branching processes is the probability of ultimate extinction, where no individuals exist after some finite number of generations(sub-critical case). The ultimate extinction probability is given by

$$\lim_{n\to\infty} P(Z_n = 0)$$

In addition to the extinction case, we have 3 different cases in total. We denote the total population of generation n as E(X_n).

(1) Sub-critical case: E(X_n)$\to$0 as n$\to \infty$, that is the specious is tending to extinction after several generations.
(2) Critical case: E(X_n)=1 for all n. That is the total population of this specious is constantly 1.
(3) Supercritical case: E(X_n) $\to \infty$ as n $\to \infty$. That is the specious will never die out.

In this project, we shall focus on the sub-critical case.

1.2. Notations.

Recall the notation from Chapter 2, and the property of Markov Chain,we have :

$$P(Z_{n+1} = Z_{n+1}|Z_n = k) = P(X_{n,1} + X_{n,2} + X_{n,3} + ... + X_{n,k} = Z_{n+1})$$

and we denote the π_0 as the probability of processes die out as:

$$\pi_0 = P(Z_n = 0|n \in \mathbb{Z}) or \lim_{n\to\infty} P(Z_n = 0)$$

The probability density function for offspring is often called the $offspring\ distribution$ and denoted as

$$p_m = P(X_{n,i} = m)$$

Obviously, π_0=0 if p_0=0. We denote μ as the expectation of the reproduction distribution of a individual.

EXAMPLE 3.1. The expectation μ of the generation 1 shall be express as :

$$\mu = E(Z_1) = \sum_{k=0}^{\infty} kp_k$$

1.3. Mathematically Define the Branching Processes.

THEOREM 3.2. *Denote m_n=E(Z_n) then we have*

$$m_{n+1} = m_n\mu$$

PROOF. Since the expectation of all the individuals are identical, the expectation of descendant number of each individual should always be μ. Also, considering we have the population expectation m_n in nth generation, so the expectation of total population in generation $n+1$ is m_{n+1}=m_n μ. □

THEOREM 3.3. *As n→ ∞, we have the mathematical expression of the three different cases:*

(1) *Sub-critical case: As n → ∞, if μ <1 then we have π_0=1.(Specious is tending to extinction after several generations.)*
(2) *Critical case: As n → ∞,if μ =1 then we have π_0=0. (The population is constantly 1. This case is an extremely specific case, it's probability is zero.)*
(3) *Supercritical case: As n → ∞, if μ >1 then we have π_0=0.(That is the specious will never die out.)*

PROOF. Recall the Example 3.2 and Theorem 3.3, we have m_n=μ^n. So we have:

(1) Sub-critical case: As n → ∞, if $\mu < 1$ then $\mu^n \to 0$. That is π_0=1. The expectation of specious' population after n generations will become 0.
(2) Critical case: As n → ∞,if $\mu < 1$ then μ^n=1^n=1. Then we have π_0=0, and the expectation of specious' population is constantly 1
(3) Supercritical case: As n → ∞, if μ >1 then $\mu^n > 1$. Then we have π_0=0, that is the expectation of specious' population will always greater than 1.

□

2. Discussion of Moment Generating Function

In this section we will have some discussion of moment generating function of reproduction distribution. First of all we denote ϕ as the moment generation function and s as e^t.

$$\phi(s) = \phi_{Z_1}(s) = E(s^{Z_1}) = \sum_{k=0}^{\infty} p_k s^k$$

and for generation n, and Z_n we have

$$\phi(s) = \phi_{Z_n}(s) = E(s^{Z_n}) = \sum_{k=0}^{\infty} P(Z_n = k) s^k$$

To study the convergence of the power series above, we will assume the 0$\leq$ s $\leq$.

Now let's try to find the recursive equation for ϕ_{Z_n}

$$\phi_{Z_n}(s) = E[s^{Z_n}]$$

$$\phi_{Z_n}(s) = \sum_{k=0}^{\infty} E[s^{Z_n} | Z_{n-1} = k] P(Z_{n-1} = k)$$

$$\phi_{Z_n}(s) = \sum_{k=0}^{\infty} (E(s^{X_1})(E(s^{X_2})...(E(s^{X_k}))P(Z_{n-1} = k)$$

$$\phi_{Z_n}(s) = \sum_{k=0}^{\infty} \phi(s)^k P(Z_{n-1} = k)$$

$$\phi_{Z_n}(s) = \phi_{Z_{n-1}}(\phi(s))$$

so ϕ_{Z_n}(s) is the nth iteration of ϕ.

Then we shall discuss the properties of ϕ by the recursion method again.

$$\phi(s) = \sum_{k=0}^{\infty} p_k s^k$$

$$\phi'(s) = \sum_{k=0}^{\infty} k p_k s^{k-1}$$

$$\phi''(s) = \sum_{k=0}^{\infty} k(k-1) p_k s^{k-1}$$

also, we have some special cases like:

$$\phi(0) = p_0 > 0$$

and

$$\phi(1) = \sum_{k=0}^{\infty} p_k = 1$$

and remark that

$$\phi'(1) = \sum_{k=0}^{\infty} k p_k = \mu$$

Then we shall have some analyzing of the probability generating functions. For n=3, we have:

$$\phi_{Z_3}(s) = \phi(\phi(\phi(s)))$$
$$\phi_{Z_3}(s) = \phi(\phi_{Z_2}(s))$$

so for general case, we can write:

$$\phi_{Z_n}(s) = \phi(\phi_{Z_{n-1}}(s))$$

Combining the two expression way of ϕ_{X_n}(s), we have:

$$\phi(\phi_{Z_{n-1}}(s)) = \phi_{Z_{n-1}}(\phi(s))$$

as $n \to \infty$, the extinction case can be explained as ϕ_{Z_n}(s) $=$ $\phi_{Z_{n-1}}$(s), then we can say

$$\phi(\phi_{Z_{n-1}}(s) = \phi_{Z_{n-1}}(s)$$

and that is the solution of ϕ(s) $= s$ is the probability of extinction. Notice that ϕ'(1)=μ so we have

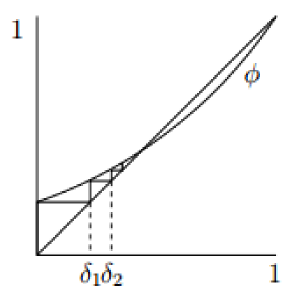
$$\phi''(s) = \sum_{k=0}^{\infty} k(k-1)p_k s^{k-1} \geq 0$$

Denote δ_n as the probability that the specious be extinct at generation n, then we have:

$$\delta_n = \phi_{X_n}(0)$$

and so it is obtained by starting at 0 and computing the nth iteration of ϕ. We know that δ_n is a nondecreasing sequence(If Z_{n-1}=0, than we must have Z_n=0).

Now we shall discuss the probability of extinction in 2 cases: $\mu \leq 1$, and $\mu > 1$. As the analyzing of derivatives of $\phi_{X_n}(0)$ and the Theorem 3.4, we know that for $\mu \leq 1$, δ_n converges to 1(in other words, $\pi_0 = 0$). On the other hand, from Equation 2.6 we can have that s=1 is always an solution of functionϕ(s) $= s$, so in this case there is exactly one solution lies in the interval (0, 1) as the figure below shows:



Conclude all researches above, we have the following Theorem.

Theorem 3.4. *If $\mu > 1$, then π_0 is the smallest solution on [0,1] to s=ϕ(s).*

Example 3.5. If we have Z_0=k instead, how will the properties above change?

Solution 4. We can consider the k individuals at the beginning generation separately. Since for each individual, the probability of extinction is identically π_0, so the probability of whole specious dies out is π_0^k

Example 3.6. Assume that the reproduction distribution of a specious follow the Poisson Distribution with parameter λ, then the probability of extinction π_0 should be the solution of what function?.

Solution 5. As the definition of Poisson Distribution, we have:

$$p_k = \frac{\lambda^x}{x!}e^{-\lambda}$$

so we have

$$\phi(s) = \sum_{k=0}^{\infty}(s^k)\frac{\lambda^x}{x!}e^{-\lambda}$$

so the required function is

$$s = \sum_{k=0}^{\infty}(s^k)\frac{\lambda^x}{x!}e^{-\lambda}$$

CHAPTER 4

The Application of Branching Process in Macro-economics

Before we start the application, there are several concepts to be declare.

1. Fiscal multiplier

This section is devoted to review some basic concepts and history of fiscal multiplier which will used throughout this project.

As the world is in a period of economic contraction (or says the recovery period of recession), most of the governments around the world will sketch expansion fiscal policy to motivate the market as well as increasing the money liquidity. To achieve this propose, what the Canadian government did was claimed an additional government spending on public infrastructure. To understand the exact effect of the extra spending to economy, we shall invite the concepts of $marginal\ propensity\ to\ consume$($MPC$) and $fiscal$ $multiplier$.

1.1. Marginal Propensity to Consume.

Definition 4.1. $Marginal\ propensity\ to\ consume = \frac{extra\ consume}{extra\ income}$. It is the proportion of disposable income which individuals spend on consumption is known as propensity to consume. MPC is the proportion of additional income that an individual consumes.

The $Marginal\ Propensity\ to\ Consume$ is a metric that quantifies induced consumption. The concept comes from Keynes consuming theory. The multiplier emerged from arguments in the 1920s and 1930s over how governments should respond to economic slumps. The concept of fiscal multiplier was first mentioned in the British economist Hahn's essay *The relationship between domastic investment and unemployment*,which was published on the *Econimist Journal*, 1931 *June*.

After that, John Maynard Keynes, one of history's most important economists, described the role of the multiplier in detail in his seminal book, "The General Theory of Employment, Interest and Money". In 1933 April, Keynes gave out the function to calculate the multiplier in his essay *Multiplier theory* . As this theory, the MPC should always lies in the interval [0,1]. As the former example we mentioned, the additional government spending in construction industry will increase the income of a builder's income by 100 dollars, and he decides to spend 30 dollars to buy a pair of new boots, that is the MPC equals to $\frac{30}{100}$.

1.2. Fiscal Multiplier.

Definition 4.2. A *fiscal multiplier* we are using in this project is a ratio of change to national consuming to government's extra spending to cause it.

Example 4.3. To understand this concept better, we will give an example from individual version first. As the former paragraph mentioned, the government's additional spending of particular industry will increase the income of an employee in this industry first. In macro-economy, the additional income will always trigger the extra consuming, and then generates new income to people in other industries (i.e. extra consuming in the restaurant will generate extra income to the waiters) as the second round extra income. The activity will keep going as the process of recursion(i.e. the waiter with extra income may buy a new television, which will increase the income of worker in electronic industry), and in economy, the effect on total income or says "aggregate demand" through this expansion fiscal policy is called fiscal multiplier.

1.3. What's the same between Branching Processes and Multiplier's Estimation.

In Fiscal Multiplier model, the case is, if the government gives 100 dollars to a people, this individual then will have 100 dollars additional income. All of them has a probability P to spend a fixed proportion of this 100 dollars for extra consuming or (1-P) probability to deposit this 100 dollars into his account. After the first round excess consuming, the amount of money they used will become the second round additional income. And the people who hold the second round additional income will have the same binomial distribution of whether deposit the money or use it for extra consuming. We notice that there is an important assumption that every consumer will have the same probability of taking extra consuming and every consumer has same Marginal Propensity Consume. Also, these assumptions are exactly same as the assumptions in Branching processes that all the individuals have same spawning ability within the same environment.

2. What's New in My Model

The traditional method of fiscal multiplier estimation assume that all the consumers will definitely generate the extra consuming. However, this may not happen in the real economy. In my model, each individual will have a probability P to generate extra consuming with their extra income. By introducing the concept of p, the estimation of fiscal multiplier will be more accurate, and we can then discuss the minimum value of p so that the multiplier's impact won't die out before a given generations consuming.

2.1. Mathematical Explanation.

As the former subsection mentioned, each individuals has the same probability p to generate the extra consuming, so the offspring distribution is binomial. We assume there will be 3 generations extra consuming at most in one year. Then the offspring distribution is B(3,p). And we need to compute π_0. As $\mu = 3p$, π_0 is the solution of function ϕ(s)=s in interval (0,1).

$$\phi(s) = \sum_{k=0}^{3} \binom{3}{k} p^k (1-p)^{3-k} s^k = s$$

since we know that s=1 is always a solution of this function, the function can be rewrite as:

$$p^3 s^2 + (3p^2 - 2p^3)s + (p-1)^3 = 0$$

denote this function as h(s)

Obviously, we have

$$h(0) = (p-1)^3 < 0 \text{ (as } P \leq 1) \text{ and}$$
$$h(1) = p^3 + (3p^2 - 2p^3) + (p-1)^3 = 3p - 1$$

Recall Theorem 3.4 in Chapter 3, we now that if the number of consumers' generation tending to infinity, then μ has to be greater than 1 to ensure that the multiplier effect won't die out, and that is $p > \frac{1}{3}$.

In addition, since this is a quadratic function of s, and $P > 0$, so whenever $3p - 1 > 0$, there is always a solution lies in (0,1). And through the general solution of quadratic function, we have:

$$s = \frac{(2p-3)p \pm \sqrt[2]{p(4-3p)}}{2p^2}$$

(2.1)

3. Additional Notes

To test our research, we assume the offspring distribution is B(3, 0.5). Then our extinction function is:

$$\phi(s) = \frac{1}{8}s^3 + \frac{3}{8}s^2 + \frac{3}{8}s + \frac{1}{8} = s$$

through the numerical method calculation, we can find out that the function has 3 solutions: $s = 1$, $s = -\sqrt{5}$ - 2 and $s = \sqrt{5}$ - 2. Only $s = \sqrt{5}$ - 2 lies in the interval (0,1), so the extinction probability π_0 of this case is $\sqrt{5}$ - 2 (0.2361).

The statement of real case may be: The government is planing to have a year-round economic stimulation project, the estimated extra consuming is 3 times during the period. Please identify to critical probability of each individual consumers' extra consuming willing, and analyze that the interval of probability p that we can make sure the extinct probability is no larger than 5% ?

From the research of previous section, we know that the critical probability is defined by $\mu > 1$, since $\mu =$ np and n = 3, so critical probability p is $\frac{1}{3}$. And analyze the inequality

$$\frac{(2p-3)p \pm \sqrt[2]{p(4-3p)}}{2p^2} < 0.05$$

(3.1)

can help us to define the interval of probability p.

Bibliography

[1] Xianping Li, "The Fundamental Probability Theory" Higher Education Press, 2010.

[2] W. Feller, "An introduction to probability theory and its applications", 1–2, Wiley (1950–1966)

[3] Huilai Li, "Probability and Mathematical Statistics" Tsinghua University Press, 2005.

[4] "Discrete Stochastic Processes" R.G Gallager 2009.

[5] J. Susan Milton, Jesse C. Arnold, ”Introduction to Probability and Statistics: Principles and Applications for Engineering and the Computing Sciences 4th Edition”, The McGraw-Hill Companies,Inc.2003.

[6] N. Gregory Mankiw, ”Principles of Macroeconomics 6th Canadian Edition”,Nelson Education Ltd, 2008