JUNIOR TEAM - SOLUTIONS

1.

$3 = (4 + 4 + 4) \div 4$

$4 = \sqrt{4 + 4 + 4 + 4}$

$5 = (4 \times 4 + 4) \div 4$

$6 = 4! \times 4 \div (4 \times 4)$

$7 = 4 + 4 - (4 \div 4)$

$8 = 4 + 4 + 4 - 4$

$9 = 4 + 4 + (4 \div 4)$

$10 = (44 - 4) \div 4$

$11 = 44 \div \sqrt{4 \times 4}$

$12 = 4 \times 4 - \sqrt{4 \times 4}$

2. Write one million in base 7. We get $7^7 + 7^6 + 3(7^5) + 3(7^4) + 3(7^3) + 3(7^2) + 7^1 + 7^0$.

3. The trains close the distance at 160 km/h, so they will meet in 3 hours. If the fly travels 140 km/h, it must travel 420 km.

4. For each n, $1^n = 1$ and 4^n is divisible by 4, so we get a remainder of 1for those two terms. When $n = 1$, we get 10, which has a remainder of 2. Otherwise, 4 divides 2^n, so we have only to examine 3^n. But here we have $3^n = (4 - 1)^n = 4^n - \ldots + (-1)^n$, and each term except the last is divisible by 4, so we get a remainder of $(-1)^n$ here, giving a total remainder of 2 when n is even and zero when n is odd.

5. Let us say we have $AB + CD = EFG$. Evidently $E = 1$. Now $A \neq 0$ and $C \neq 0$. If $B = 0$ or $D = 0$, then $G = D$ or B respectively, which is impossible. Thus, $F = 0$ or $G = 0$. If $G = 0$, then $\{B,D\} = \{4,6\}$. But then $A + C \leq 3 + 5$, contradicting the fact that the sum is a 3-digit number. Thus $F = 0$. Therefore, $A + C = 9$ or 10. But if it is 9, then $\{A, C\} = \{3,6\}$ or $\{4,5\}$. In the first case, $B + D \leq 6 + 3$, so there is no carry, and this is a contradiction. So $A + C = 10$. But then $\{A,C\} = \{4,6\}$. Thus $\{B,D,G\} = \{2,3,5\}$, and as $B + D = G + 10$, we can only have $G = 5$.

6. Let x be the number thought of by the person who announced 4. Then if a is the number thought of by the person who announced 2, then the average of x and a must be 3, so $a = 6 - x$. If b is the number thought of by the person who announced 5, then the average of a and b is 1, so $b = 2 - a = x - 4$. If c is the number thought of by the person who announced 3, then the average of b and c is 4, so $c = 8 - b = 12 - x$. If d is the number thought of by the person who announced 1, then the average of c and d is 2, so $d = 4 - c = x - 8$. But also, the average of x and d is 5, so $d = 10 - x$. Thus, $x - 8 = 10 - x$, and $x = 9$.

7. Anna and Bob go across, Anna comes back. Carol and Duke go across, Bob comes back. Anna and Bob go across. (2 + 1 + 10 + 2 + 2 = 17 minutes).

8. Let a be the length of any one side of the hexagon. Joining diagonally opposite vertices in the hexagon, we can divide it into 6 equilateral triangles with side length a. Now, since the triangle has the same perimeter, each side has length $2a$. Joining the midpoints of the three sides together, we divide the triangle into 4 equilateral triangles with side length a. Thus, the hexagon has area equal to 6/4 times that of the triangle, or 15.

9. No pure mathematician, whether well or unwell, would claim to be unwell. Therefore Wanda is applied. Similarly, Zeke is pure. Nobody who is well would claim to be an applied mathematician, so Yolanda is unwell. Similarly, Xavier is well. Since Zeke is pure, he believes that Yolanda is well, but she isn't. Therefore Zeke is unwell. Xavier is well, and has made a true statement about Zeke, so Xavier must be pure. Yolanda is unwell and has made a false statement and Xavier. Therefore she must be pure. Wanda is applied and has made a true statement about Yolanda, so she must be unwell.

10. For convenience, let P be the origin, and say the circle is $(x - a)^2 + (y - b)^2 = r^2$. If (x, y) is any point on the circle, then the midpoint in question is $(x/2, y/2)$. Notice that $(x/2-a/2)^2+(y/2-b/2)^2 = 1/4\ ((x-a)^2+(y-b)^2) = (r/2)^2$, so the midpoints lie on a circle centered at $(a/2, b/2)$ with radius $r/2$.