The bounded Lie Engel property on torsion group algebras

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Abstract

Let F be a field of characteristic different from 2 and G a group with involution $*$. Extend the involution to the group ring FG, and write $(FG)^-$ for the Lie subalgebra of FG consisting of the skew elements. We classify the torsion groups G having no elements of order 2 such that $(FG)^-$ is bounded Lie Engel.

1 Introduction

Let R be a ring with involution $*$. Write R^+ for the set of symmetric elements and R^- for the set of skew elements. That is, $R^+ = \{r \in R : r^* = r\}$ and $R^- = \{r \in R : r^* = -r\}$. It is an interesting problem to discover the extent to which R^+ and R^- determine the structure of R.

In particular, let G be a group equipped with an involution $*$, and let F be a field of characteristic different from 2. Extending $*$ linearly to the group ring FG, we notice that $(FG)^+$ is the set of linear combinations of terms of the form $g + g^*$, $g \in G$, and $(FG)^-$ is the set of linear combinations of terms of the form $g - g^*$. We define the Lie product on FG via

$$[x_1, x_2] = x_1x_2 - x_2x_1$$

and

$$[x_1, \dots, x_{n+1}] = [[x_1, \dots, x_n], x_{n+1}].$$

A subset S of FG is said to be Lie nilpotent if there exists an n such that

$$[s_1, \dots, s_n] = 0$$

for all $s_i \in S$. We say that S is Lie n-Engel if

$$[s_1, \underbrace{s_2, \dots, s_2}_{n \text{ times}}] = 0$$

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for all $s_1, s_2 \in S$, and bounded Lie Engel if it is Lie n-Engel for some n.

If $g^* = g^{-1}$ for all $g \in G$, then the induced involution on FG is called the classical involution, and a good deal is known here. Over the last two decades, a lot of attention has been devoted to determining if Lie identities satisfied by $(FG)^+$ or $(FG)^-$ are also satisfied by the whole group ring. For example, in [4], Giambruno and Sehgal showed that if $(FG)^+$ or $(FG)^-$ is Lie nilpotent, and G has no 2-elements, then FG is Lie nilpotent. Lee proved a similar result for the bounded Lie Engel property in [8]. Other results have been proved for groups with 2-elements, and we refer the reader to [9] for a discussion.

More recently, however, a considerable amount of work involving other involutions on G has appeared. For example, in [7], Jespers and Ruiz Marín determined when $(FG)^+$ is commutative; subsequently, in [1], Broche Cristo, Jespers, Polcino Milies and Ruiz Marín answered the same question for $(FG)^-$. In Giambruno, Polcino Milies and Sehgal [2] and Lee, Sehgal and Spinelli [10], the conditions under which $(FG)^+$ is Lie nilpotent or bounded Lie Engel were determined for an arbitrary involution on G. In particular, if G has no elements of order 2, then the same result holds as for the classical involution. Subsequently, in [3], Giambruno, Polcino Milies and Sehgal determined when $(FG)^-$ is Lie nilpotent, if G is a torsion group having no elements of order 2. It turns out that the result is quite different here, and there are exceptional cases to consider.

Our purpose in this paper is to determine when $(FG)^-$ is bounded Lie Engel, for an arbitrary involution on G, where G is a torsion group without 2-elements. Our main result is the following.

Theorem. *Let F be a field of characteristic $p \neq 2$ and G a torsion group having no elements of order 2. Let $*$ be an arbitrary involution on G, and extend it F-linearly to FG. Then $(FG)^-$ is bounded Lie Engel if and only if either FG is bounded Lie Engel or $p > 2$, FG satisfies a polynomial identity and G has a normal $*$-invariant p-subgroup N of bounded exponent such that the induced involution on G/N is trivial.*

2 Preliminary matters

Let us gather some necessary results. First of all, the conditions under which a group ring is bounded Lie Engel were determined by Sehgal in [14, Theorem V.6.1]. Recall that a group G is said to be p-abelian if G' is a finite p-group and 0-abelian means abelian.

Lemma 1. *Let F be a field of characteristic $p \geq 0$ and G a group. If $p = 0$, then FG is bounded Lie Engel if and only if G is abelian. If $p > 0$, then FG is bounded Lie Engel if and only if G is nilpotent and G contains a p-abelian normal subgroup of p-power index.*

We also need the result on skew elements with respect to the classical involution due to Lee.

Lemma 2. *Let char F be different from 2 and let G be any group without 2-elements. If the skew elements of FG with respect to the classical involution are bounded Lie Engel, then FG is bounded Lie Engel.*

Proof. See [8, Theorem 3]. □

Recall that if R is an F-algebra, then R is said to satisfy a polynomial identity if there exists a nonzero polynomial $f(x_1, \ldots, x_n)$ in the free algebra $F\{x_1, x_2, \ldots\}$ such that $f(r_1, \ldots, r_n) = 0$ for all $r_i \in R$. The conditions under which FG satisfies a polynomial identity were determined by Isaacs and Passman.

Lemma 3. *Let F be a field of characteristic $p \geq 0$ and G a group. Then FG satisfies a polynomial identity if and only if G has a p-abelian normal subgroup of finite index.*

Proof. See [11, Corollaries 5.3.8 and 5.3.10]. □

Now, if R is an F-algebra with involution, then we say that R satisfies a $*$-polynomial identity if there exists a nonzero polynomial $f(x_1, x_1^*, \ldots, x_n, x_n^*)$ in the free algebra with involution $F\{x_1, x_1^*, \ldots\}$ such that $f(r_1, r_1^*, \ldots, r_n, r_n^*) = 0$ for all $r_i \in R$. In particular, if $(FG)^-$ is Lie n-Engel, then FG satisfies

$$[x_1 - x_1^*, \underbrace{x_2 - x_2^*, \ldots, x_2 - x_2^*}_{n \text{ times}}].$$

By a classical result due to Amitsur (see [6, p. 196]), if R satisfies a $*$-polynomial identity, then R satisfies a polynomial identity. Thus, in view of the preceding lemma, if G is torsion and $(FG)^-$ is bounded Lie Engel, then G is locally finite. We assume that fact throughout the paper without further mention.

Not surprisingly, some of the arguments from [3] for the Lie nilpotent skew elements also apply here. Let us extract some facts.

Lemma 4. *Let R be a semiprime ring with involution, with $2R = R$. If R^- is bounded Lie Engel, then the skew elements commute and R satisfies the standard polynomial identity on 4 variables, St_4.*

Proof. The proof is essentially identical to that of [2, Lemma 2.4] (which gave the same result for R^+), and so is left to the reader. □

From this point on, we let G be a torsion group without 2-elements having an involution $*$ extended linearly to an involution on FG.

Lemma 5. *Suppose char $F \neq 2$ and FG is semiprime. If $(FG)^-$ is bounded Lie Engel, then G is abelian.*

Proof. In view of Lemma 4, follow the argument after Theorem 2.3 of [3] verbatim. □

In particular, we recall Passman's result (see [11, Theorems 4.2.12 and 4.2.13]) that if char $F = p \geq 0$, then FG is semiprime if and only if G has no finite normal subgroups with order divisible by p. Thus, the characteristic zero case is done, and throughout, we let char $F = p > 2$.

Lemma 6. *If $(FG)^-$ is bounded Lie Engel, then the p-elements of G form a subgroup P and G/P is abelian. Furthermore, if the induced involution on G/P is not trivial, then $G = P \times Q$, where Q is an abelian p'-group.*

Proof. Follow the proof of [3, Lemma 2.5] verbatim in order to show that P is a subgroup. As $F(G/P)$ is semiprime, it follows from Lemma 5 that G/P is abelian.

Suppose that $*$ is not trivial on G/P. In order to show that $G = P \times Q$, it suffices to show that the p'-elements of G form an abelian group. It is sufficient to consider finite subgroups of G. Indeed, if H is a finite subgroup of G, then choose $x \in G$ such that $x^* \notin xP$. Let K be the finite subgroup of G generated by H, H^*, x and x^*. Then K is $*$-invariant. It suffices to prove that K is nilpotent, for then its p'-elements do indeed form a group, and since the corresponding group ring is semiprime, we see from Lemma 5 that the p'-elements commute. To this end, apply the proof of [3, Theorem 3.2], replacing the reference to [4, Theorem] with an appeal to Lemmas 1 and 2. $\square$

Two more lemmas will be useful. The first is [2, Corollary 2.10].

Lemma 7. *Let A be an abelian torsion group with no 2-elements having an involution $*$. Then $A = A_1 \times A_2$, where A_1 is the set of symmetric elements of A and A_2 is the subset of A upon which $*$ acts as the classical involution.*

We use the notation A_1 and A_2 throughout the paper.

Lemma 8. *Suppose that G has a $*$-invariant abelian normal subgroup A. Then*

1. *if $x \in G \backslash A$ satisfies $x^* \in x^{-1}A$, then there exists $c \in A_1$ such that $(xc)^* = (xc)^{-1}$ and*
2. *if A is finite, $x \in G$ satisfies $x^* \in xA$, and $(o(x), |A|) = 1$, then there exists $c \in A_1$ such that $(xc)^* = xc$.*

Proof. The first part is [3, Lemma 2.8]. For the second part, notice that $\langle A, x\rangle$ is $*$-invariant, and furthermore, $\langle A, x\rangle = A \rtimes \langle x\rangle$. Now combine Lemma 2.8 and Remark 3.1 in [3]. $\square$

3 Proof of the main result

Throughout, let G be a torsion group without 2-elements. Let F be a field of characteristic $p > 2$, and suppose that FG has an involution induced from one on G. Write P for the set of p-elements of G. In view of Lemma 6, if $(FG)^-$ is bounded Lie Engel, then P is a (normal $*$-invariant) subgroup of G. Let us first dispense with the case where $*$ is not trivial on G/P. We begin with the following lemma (which still holds if G has 2-elements).

Lemma 9. *Let G be any torsion group such that the skew (or symmetric) elements of FG are Lie p^n-Engel. If G has a central subgroup H of unbounded exponent such that $*$ acts as the classical involution on H, then FG is Lie p^n-Engel.*

Proof. We will prove the result for the skew elements; the proof for the symmetric elements is the same, *mutatis mutandis.* As observed in the proof of [4, Theorem 2], since the $*$-polynomial identity

$$[x_1 - x_1^*, \underbrace{x_2 - x_2^*, \ldots, x_2 - x_2^*}_{p^n \text{ times}}]$$

is linear in x_1, FG satisfies

$$[x_1, \underbrace{x_2 - x_2^*, \ldots, x_2 - x_2^*}_{p^n \text{ times}}] = [x_1, (x_2 - x_2^*)^{p^n}].$$

Take any $\alpha, \beta \in FG$, and $z \in H$. Then $[\alpha, (z\beta - z^{-1}\beta^*)^{p^n}] = 0$. Now, let K be the $*$-invariant subgroup of G generated by the supports of α and β and let $L = KH$. Then K is finite and normal in L. Furthermore, since H is unbounded, we can choose z in such a way that the cosets $z^i K$ are distinct, $-p^n \leq i \leq p^n$. But then looking at the $z^{p^n}K$ coset, we see that the only part that occurs is $[\alpha, (z\beta)^{p^n}]$. Thus,

$$0 = [\alpha, \beta^{p^n}] = [\alpha, \underbrace{\beta, \ldots, \beta}_{p^n \text{ times}}],$$

as required. □

The following lemma deals with p-groups. As usual, we write $(g, h) = g^{-1}h^{-1}gh$ and $\zeta(G)$ for the centre of G.

Lemma 10. *Let P be a p-group. If $(FP)^-$ is bounded Lie Engel, then P is nilpotent.*

Proof. We know that FP satisfies a polynomial identity, hence P has a p-abelian normal subgroup A of finite index. Replacing A with $A \cap A^*$, let us assume that A is $*$-invariant.

Since A' is finite, we know that A is nilpotent (see [9, Lemma 4.3.12]). Thus, by Hall's criterion (see [13, 5.2.10]), it is sufficient to show that P/A' is nilpotent. We therefore replace P with P/A' and assume that A is abelian.

Let us now proceed by induction on the index, $(P : A)$. If $P = A$, there is nothing to do. Otherwise, P/A, being a finite p-group, has a nontrivial centre, H/A. Since H/A is abelian, we write $H/A = (H/A)_1 \times (H/A)_2$, and therefore either $(H/A)_1$ or $(H/A)_2$ is nontrivial. That is, there exists $x \in H$ such that $o(xA) = p$ and $(xA)^* = x^{\pm 1}A$. Let $L = \langle A, x \rangle$. Then L is a $*$-invariant normal subgroup of P. If we can show that L is nilpotent, then by Hall's criterion, it suffices to show that P/L' is nilpotent. But P/L' has the abelian normal subgroup L/L', and $(P/L' : L/L') = (P : L) < (P : A)$. Thus, by our inductive hypothesis, P/L' is indeed nilpotent. Therefore, we need only show that L is nilpotent. In particular, we may assume that $P/A = \langle xA \rangle$, where $o(xA) = p$ and $(xA)^* = x^{\pm 1}A$.

Next, we notice that $P' = \{(a, x) : a \in A\}$ (see, for instance, [9, Lemma 1.3.4]). If, for every $a \in A$, we have $(a, x)^{p^n} = 1$, then since A is abelian, $a^{-p^n}(a^x)^{p^n} = 1$, hence $(a^{p^n}, x) = 1$. In particular, for any $h \in P$, $h^p \in A$, hence $h^{p^{n+1}}$ centralizes both A and x. Thus, $P/\zeta(P)$ is a p-group of bounded exponent. By [9, Lemma 3.2.7], $P/\zeta(P)$ is nilpotent and therefore, so is P. Thus, it suffices to show that P' has bounded exponent.

Now, we know that $x^* = x^{\pm 1}c$ for some $c \in A$. Let K be the normal closure in P of $\langle c, c^* \rangle$. Since $c, c^* \in A$, and A is an abelian subgroup of finite index, it follows easily that K is finite. If we can show that $(P/K)'$ has bounded exponent, then surely P' does as well. Thus, we factor out K and assume that $x^* = x^{\pm 1}$. In a similar manner, we factor out $\langle x^p \rangle$ and assume that $x^p = 1$. Thus, $P = A \rtimes \langle x \rangle$.

Write $A = A_1 \times A_2$. In view of our reductions, we can now follow the proof of [3, Proposition 4.1] to see that if $(FP)^-$ is Lie p^m-Engel, then (after possibly factoring out another finite normal $*$-invariant subgroup), $A_2^{p^m}$ is central. If A_2 is of unbounded exponent then so is $A_2^{p^m}$, hence, by the preceding lemma, FP is Lie p^m-Engel. It now follows from Lemma 1 that P is nilpotent. Since FP satisfies

$$0 = [x_1, \underbrace{x_2, \ldots, x_2}_{p^m \text{ times}}] = [x_1, x_2^{p^m}],$$

we see that $P/\zeta(P)$ has bounded exponent. But now by [14, Corollary I.4.3], P' has bounded exponent, as required. Assume that A_2 is bounded. Then the normal $*$-invariant subgroup it generates, being contained in A, is also bounded. Call it N. If $(P/N)'$ is bounded, then surely so is P'. Thus, we may factor out N and assume that every element of A is symmetric.

Now, if $x^* = x$, then for every $a \in A$ we have $a^x \in A$, hence $x^{-1}ax = (x^{-1}ax)^* = xax^{-1}$, and therefore $x^2a = ax^2$. Since x has odd order, $xa = ax$. Thus, A is central, so P is abelian.

We therefore assume that $x^* = x^{-1}$. Now, for any $\alpha, \beta \in (FP)^-$, we have $[\alpha, \beta^{p^m}] = 0$. If some $\beta^{p^m} \neq 0$, then since $\beta^{p^m} \in (FP)^-$, we see that $(FP)^-$ has a nonzero centre. The last part of the proof of Proposition 4.1 in [3] now applies, and we find that P' is finite in this case. Thus, we assume that $\beta^{p^m} = 0$ for all $\beta \in (FP)^-$.

In order to deal with this final case, we borrow some constructions from [12]. For any $\delta \in FA$, we define the trace map

$$\operatorname{tr}(\delta) = \sum_{i=0}^{p-1} \delta^{x^i}.$$

We note that $\operatorname{tr}(\delta)$ is always central in FP, and furthermore, letting $\tau = \sum_{i=0}^{p-1} x^i$, we see that $\tau\delta\tau = \operatorname{tr}(\delta)\tau$. Also, τ is clearly symmetric. Thus, for

all $a \in A$,
$$0 = (a\tau - (a\tau)^*)^{p^m} = (a\tau - \tau a)^{p^m}.$$
But notice that
$$(a\tau - \tau a)^2 = a\tau a\tau - a\tau^2 a - \tau a^2\tau + \tau a\tau a.$$
Since $\tau^2 = 0$, we get
$$(a\tau - \tau a)^2 = a\mathrm{tr}(a)\tau - \mathrm{tr}(a^2)\tau + \mathrm{tr}(a)\tau a.$$
Next,
$$(a\tau - \tau a)^3 = a\tau a\mathrm{tr}(a)\tau - \tau a^2\mathrm{tr}(a)\tau - a\tau\mathrm{tr}(a^2)\tau + \tau a\mathrm{tr}(a^2)\tau + a\tau\mathrm{tr}(a)\tau a - \tau a\mathrm{tr}(a)\tau a.$$
Since traces are central, two of the terms contain τ^2 and vanish. We are left with
$$\mathrm{tr}(a)a\mathrm{tr}(a)\tau - \mathrm{tr}(a)\mathrm{tr}(a^2)\tau + \mathrm{tr}(a^2)\mathrm{tr}(a)\tau - \mathrm{tr}(a)\mathrm{tr}(a)\tau a = \mathrm{tr}(a)^2(a\tau - \tau a).$$

Thus, every odd power of $a\tau - \tau a$ is just going to give us $\mathrm{tr}(a)^i(a\tau - \tau a)$, for some i. That is,
$$\mathrm{tr}(a)^i a\tau = \mathrm{tr}(a)^i \tau a.$$
Multiplying by τ on the right, we get
$$0 = \mathrm{tr}(a)^i \tau a\tau = \mathrm{tr}(a)^{i+1}\tau.$$
Since $\mathrm{tr}(a) \in FA$, this can only mean that $\mathrm{tr}(a)^{i+1} = 0$. Thus, for some suitable r, $\mathrm{tr}(a)^{p^r} = 0$ for all $a \in A$. But $\mathrm{tr}(a)^{p^r} = \sum_{i=0}^{p-1}(a^{p^r})^{x^i}$. Now, if a sum of p group elements comes to 0, then they are all equal. That is, $a^{p^r} = (a^{p^r})^x$, so a^{p^r} is central. We conclude that P has bounded exponent modulo its centre, hence by [14, Corollary I.4.3], P' has bounded exponent. We are done. □

We now have the following

Proposition. *If $(FG)^-$ is bounded Lie Engel, and $*$ is not trivial on G/P, then FG is bounded Lie Engel.*

Proof. By Lemma 6, we know that $G = P \times Q$, where Q is abelian. Furthermore, since FP satisfies a polynomial identity, P has a p-abelian normal subgroup of finite index. The preceding lemma tells us that P (and hence G) is nilpotent. In view of Lemma 1, we are done. □

Thus, we may assume that the induced involution is trivial on G/P. We now have

Lemma 11. *If the skew elements of FG are bounded Lie Engel but FG is not, then $(P/P')_2$ has bounded exponent.*

Proof. Notice that if, for a particular G, we find that $F(G/P')$ is bounded Lie Engel, then G/P' is nilpotent. We already know from Lemma 10 that P is nilpotent. Thus, we find that G is nilpotent, hence $G = P \times Q$ where Q is a p'-group. Since G/P is abelian, Q is abelian. Now, since FP satisfies a polynomial identity, we know that P has a p-abelian normal subgroup of finite index. We conclude from Lemma 1 that FG is bounded Lie Engel, giving us a contradiction.

Thus, we replace G with G/P' and assume that P is abelian, with P_2 unbounded. Let $(FG)^-$ be Lie p^m-Engel. We claim that $P_2^{p^m}$ is central. Take $a \in P_2$, $x \in G$. Write $x = yz$ where y and z are powers of x such that $y \in P$ and z is p'. Since P is abelian, a^{p^m} commutes with y, so it suffices to show that a^{p^m} commutes with z. That is, we assume that x is a p'-element. Now, $\langle x, x^* \rangle$ is finite, and its p-elements form a normal subgroup C. By Lemma 8, since $x^* \equiv x \pmod{C}$, there exists $c \in C$ such that xc is symmetric. As it suffices to show that a^{p^m} commutes with xc, we replace x with xc and assume that x is symmetric.

Take $a, b \in P_2$. We have

$$0 = [xa - (xa)^*, (b - b^*)^{p^m}] = [xa - a^{-1}x, b^{p^m} - b^{-p^m}].$$

Noting that $(1 - a^{-x}a^{-1})a$ commutes with b, we get

$$0 = [x, b^{p^m} - b^{-p^m}](1 - a^{-x}a^{-1})a.$$

Now, if there are infinitely many distinct elements $a^{-x}a^{-1}$ with $a \in P_2$, then we must have $[x, b^{p^m} - b^{-p^m}] = 0$, hence $xb^{p^m} = b^{p^m}x$ or xb^{-p^m}. In the latter case, $b^{2p^m} = 1$, hence $b^{p^m} = 1$ and so x commutes with b^{p^m} in any case.

So, suppose that there are only finitely many elements $a^{-x}a^{-1}$, $a \in P_2$. Let $L = \langle P, x \rangle$. Since P is an abelian normal subgroup of finite index, P is contained in the FC-centre of L. Thus, let N be the finite $*$-invariant normal subgroup of L generated by the $a^{-x}a^{-1}$. Let $\bar{L} = L/N$. Then for all $a \in P_2$, $(\bar{a})^{-\bar{x}} = \bar{a}$. That is, $(\bar{a})^{\bar{x}^2} = \bar{a}$. Since $\bar{x}^2$ has odd order, $\bar{a}$ commutes with $\bar{x}$. Thus, $\langle \bar{P}_2, \bar{x} \rangle$ is abelian. Therefore, letting $H = \langle P_2, x \rangle$, we see that H' is a finite p-group. We now borrow a construction from [3]. Let $B = H' \cap \zeta(H)$ and let A be the second centre of H. Certainly B is a finite central $*$-invariant p-subgroup. Furthermore, $(A, H) \subseteq H' \cap \zeta(H) = B$. Thus, A/B is a central $*$-invariant subgroup of H/B. Furthermore, by [5], since H' is finite, $(H : A) < \infty$. Since P_2 is unbounded, so are P_2B/B (since B is finite) and $(P_2B \cap A)/B$ (since A has finite index). In particular, $(A/B)_2$ is unbounded. Thus, since A/B is central, we see from Lemma 9 that $F(H/B)$ is bounded Lie Engel. Therefore, by Lemma 1, H/B is nilpotent and, since B is central, H is nilpotent. Thus, H is the direct product of a p-group (which we are assuming is abelian) and a p'-group (which must be abelian, as G' is a p-group). Therefore, H is abelian, and $(a, x) = 1$ for all $a \in P_2$.

Now, since $P_2^{p^m}$ is central, and FG is not bounded Lie Engel, we conclude from Lemma 9 that P_2 is bounded. We are done. $\square$

We can now wrap up our main result. If N is a normal subgroup of G, write $\Delta(G,N)$ for the kernel of the natural homomorphism $FG \to F(G/N)$.

Proof of Theorem. We have seen above that the $p = 0$ case is done, so assume that $p > 2$ and the skew elements are bounded Lie Engel but FG is not. We know that G/P is abelian. Furthermore, by Lemma 10, P is nilpotent. Also, by Lemma 3, P has a p-abelian normal subgroup of finite index. Therefore, by Lemma 1, FP is bounded Lie Engel. Thus, there exists an m such that $[a, b^{p^m}] = 0$ for all $a, b \in P$. In particular, $P/\zeta(P)$ is a p-group of bounded exponent. Thus, by [14, Corollary I.4.3], P' is a p-group of bounded exponent. By Lemma 11, $(P/P')_2$ is bounded. Since P/P' is abelian, it follows that the $*$-invariant normal subgroup N/P' of G/P' generated by $(P/P')_2$ is also a p-group of bounded exponent. Thus, N is a p-group of bounded exponent. Let $H = G/N$. Then the p-elements of H form a subgroup K such that every element of K is symmetric. Furthermore, we know that $*$ is trivial on H/K. Thus, if $h \in H$, let $h^* = hd$ for some $d \in K$. For any $k \in K$, we get $h^{-1}kh \in K$, hence

$$h^{-1}kh = (h^{-1}kh)^* = hdkd^{-1}h^{-1}.$$

Since d and k commute, we see that $(h^2, k) = 1$. Thus, since h has odd order, h and k commute, so K is central. But now we have

$$h = (h^*)^* = (hd)^* = h^*d^* = hd^2.$$

Therefore, $d = 1$. That is, h is symmetric for all $h \in H$. The fact that FG satisfies a polynomial identity is obvious.

Now let us check the sufficiency. If FG is bounded Lie Engel, there is nothing to say, so assume that this is not the case. Then for any $g \in G$, we have $g^* = gn$, with $n \in N$, hence $g - g^* = g(1-n) \in \Delta(G,N)$. Thus, $(FG)^- \subseteq \Delta(G,N)$. By a lemma due to Passman (see [9, Lemma 1.3.14]), we see that $\Delta(G,N)$ is nil of bounded exponent, say p^r. Therefore, $\beta^{p^r} = 0$ for all $\beta \in (FG)^-$, hence

$$[\alpha, \underbrace{\beta, \ldots, \beta}_{p^r \text{ times}}] = [\alpha, \beta^{p^r}] = 0$$

for all $\alpha, \beta \in (FG)^-$. That is, the skew elements of FG are Lie p^r-Engel. $\square$

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