Solutions

1. Alice, Bob, Carrie, David, Ellen, and Frank are going to sit in a row. Bob wants to sit adjacent to Alice, and Carrie wants to sit adjacent to David. In how many different orders they can be arranged under such conditions?

Solution: The number of ways to arrange n objects in a row is $n! = n(n-1)\dots(2)(1)$. If we had no restrictions, we would have $6! = 720$ possibilities. To deal with the restrictions, we treat Alice and Bob as one object, and similarly with Carrie and David. This gives us four objects to place in a row, so $4! = 24$ ways. For each of these however, we notice that Alice and Bob could be seated AB or BA, and Carrie and David could be seated CD or DC. This gives us $2 \times 2 = 4$ possibilities of each of the 24 ways above, so we have $4 \times 24 = 96$ possible arrangements that meet the requirements.

2. Among Aloysius, Bartholomew, Claudius, Diogenes and Ed, there is one, and only one, spy. Each person makes one statement. The spy will tell the truth, and so will exactly one other person. The following statements are given in the following order:

Aloysius: Diogenes is going to lie.
Bartholomew: Diogenes is not the spy.
Claudius: The spy is either Aloysius or Diogenes.
Diogenes: The spy has already made a statement.
Ed: I am the spy.

Who is the spy?

Solution: We first see that Diogenes cannot be the spy since if he is, he lies. This implies Bartholomew always tells the truth. If Diogenes lies, then the spy must be Ed, which also means Aloysius tells the truth. In this case, three people tell the truth. This is impossible. So Diogenes tells the truth and Bartholomew is the spy.

3. Helen has two dice. One is the standard six faces die, while the other one is a special die with two faces with "3", one face with "4", three faces with "6", and no faces with "1", "2", or "5". She throws the two dice simultaneously. What is the probability that the sum of the two faces is 8?

Solution: Each face of each die is equally likely to come up, and there are $36 = 6 \times 6$ possible combinations of two faces. To get a sum of eight, either of the two faces with 3 on the second die could be paired with the 5 on the first die, the one face with 4 on the second die could be paired with the 4 on the first die, or any of the three faces on the second die with a 6 could be paired with the 2 on the first die. This gives six ways to get a sum of 8, so the probability is $6/36 = 1/6$.

4. What is the greatest number of $90°$ interior angles that a decagon can have?

Solution: The interior angles of the polygon are all strictly between $0°$ and $360°$. Also, the sum of the interior angles is given by the formula $(n-2) \times 180°$, where n is the number of sides, so in this case we get $1440°$. These conditions rule out having 8, 9, or 10 interior angles of $90°$. It is possible to give an example with seven. Consider the polygon in the xy plane whose verticies in order are: $(3,0), (5,0), (5,3), (0,3), (0,0), (1,0), (1,1), (2,2), (4,1), (3,1), (3,0)$

Note, if we further assume that the polygon is convex, (the line segment joining two interior points lies in the interior) then we have that the exterior angles must add up to $360°$, so it is not possible to have more than three interior angles of $90°$. It is not hard to write down examples with three.

5. A square ABCD has side length 2. A semicircle is drawn inside the square so that the diameter (flat side) of the semicircle coincides with the side AB of the square. A line is drawn from the corner C of the square tangent to the semicircle, and intersecting the side AD at a point E. Find the area of the triangle CDE.

Solution: Let G be the midpoint of AB and F be the tangent point of semicircle and CE. Since CB and EA are tangent to the circle, we know EGC is a right triangle, and is similar to CGB. $|EG|/|GC| = |GB|/|CB|$, so $|EG| = |GC| \cdot |GB|/|CB| = \sqrt{5} \cdot (1/2) = \sqrt{5}/2$. Thus the area of $ABCE$ is $2 \cdot (1/2)(\sqrt{5}2)(\sqrt{5}) = 5/2$. Therefore the area of triangle CDE is $4 - 5/2 = 3/2$.

\# 6. A right triangle has an angle of $54°$. A regular polygon with n sides is positioned inside the triangle so that three of its sides lie along the sides of the triangle. What is the smallest possible value of n?

Solution: In order for three of the sides of the polygon to lie along those of the triangle, the exterior angles of the triangle must all be whole number multiples of the exterior angle of the regular polygon. The exterior angles of the triangle are $144°$, $126°$, and $90°$. Each of these must be an integer multiple of the exterior angle for the polygon, so their greatest common divisor must be an integer multiple of this angle too. The greatest common divisor of 144, 126, and 90 is 18, which corresponds to a regular polygon of 20 sides.

To see that it is possible with a 20 sided regular polygon, start with the polygon, and extend three of the sides with the right exterior angles between them to the points where they intersect to form the triangle. Then scale accordingly.

\# 7. Show that any positive integer k can be expressed as $k = \frac{mn+1}{m+n}$, where m, n are positive integers.

Solution: Let $m = 2k - 1, n = 2k + 1$. Then

$$\frac{mn + 1}{m + n} = \frac{4k^2}{4k} = k.$$

To find these m and n, one could proceed as follows:

$$mn + 1 = k(m + n)$$

$$mn - km - kn + k^2 = k^2 - 1$$

$$(m - k)(n - k) = (k - 1)(k + 1)$$

$$m - k = k - 1, n - k = k + 1.$$

\# 8. Let $A = 2222^{2221}$, B be the sum of all digits of A, C the sum of all digits of B and D the sum of all digits of C. Find D.

Solution: We know that the remainder of an integer when divided by 9 is the same as that of the sum of all its digits divided by 9. We see that $8^2 = 64 \equiv 1 \pmod 9$. Also $2222 \equiv 8 \pmod 9$, so $2222^{2221} \equiv 8^{2221} \equiv 8^{2220} \cdot 8 \equiv 1 \cdot 8 \equiv 8 \pmod 9$. So $A \equiv 8 \pmod 9$. $A < (10000)^{2221} = \left(10^4\right)^{2221} = 10^{8884}$. A has at most 8884 digits. Thus $B \leq 8884 \cdot 9 = 79956$. B is an integer which has at most 5 digits. $C \leq 7 + 4 \cdot 9 = 43$. So C contains at most 2 digits, less than 44. $D \leq 3 + 9 = 12$. But $A \equiv B \equiv C \equiv D \equiv 8 \pmod 9$. So $D = 8$.

\# 9. Let C be the unit circle $x^2 + y^2 = 1$ and P be the point $(2, 0)$. Let S denote the set of points that arise as the midpoint of a line segment from P to some point on the circle C. Determine the set S, and sketch it.

Solution: Let (a, b) be the point as the midpoint of (x, y) and $(2, 0)$. Then $a = (x + 2)/2, b = (y + 0)/2$, so $x = 2a - 2, y = 2b$. $(2a - 2)^2 + (2b)^2 = 1$ or $(a - 1)^2 + b^2 = 1/4$. This is a circle of radius $1/2$ centred at $(1, 0)$.

\# 10. Find the maximum value of $ab^2(12 - a - b)$ where $a, b \geq 0$.

Solution: We know AM-GM inequality:

$$\frac{x_1 + ... + x_n}{n} \geq (x_1 \ldots x_n)^{1/n},$$

for nonnegative sequence $x_1, \ldots, x_n$ and the equality holds if and only if $x_1 = \ldots = x_n$.
Now we see

$$ab^2(12 - a - b) = 4\left[a \cdot b/2 \cdot b/2 \cdot (12 - a - b)\right] \leq 4 \cdot \left[\frac{a + b/2 + b/2 + (12 - a - b)}{4}\right]^4 = 4 \cdot 3^4 = 324.$$

This upper bound 324 can be attained when $a = 3, b = 6$. So the maximum value is 324.

11. A pair of positive real numbers x and y satisfy $1 + \log_2 y = \log_5 x = \log_{10}(x + y)$. Find (1/x+1/y).

Solution: Let $1 + \log_2 y = \log_5 x = \log_{10}(x + y) = t$, then $y = 2^{t-1}$, $x = 5^t$ and $x + y = 10^t$. Thus

$$1/x + 1/y = (x + y)/(xy) = 10^t/[2^{t-1} \cdot 5^t] = 2.$$

12. Three smart students Adam, Ben, and Carolina are guessing two integers x and y, such that $1 < x < y < 10$. Adam is given the sum $x + y$, Ben is given the product xy and Carolina is given $(x + y) + xy$. Each student knows what information the other students have been given. The 3 students are asked to write down their answer if they know x, y for sure. But no one knows what x and y are. After knowing that no one knows the answer in the first round, all of the 3 students know the correct answer quickly in the second round. What are these two numbers x and y? Show your reasoning.

Solution: These 3 smart students cannot deduce the numbers only when the decomposition is not unique. Since Ben does not know the numbers, we know there must be at least two pairs of x and y with the same product. We just need to check 1+2+3+4+5+6+7=28 pairs, among which we find $3 \cdot 4 = 2 \cdot 6 = 12$, $3 \cdot 8 = 4 \cdot 6 = 24$ and $2 \cdot 9 = 3 \cdot 6 = 18$ are the only products with multiple decompositions. Carolina has the sum $x + y + xy = (x + 1)(y + 1) - 1$. The only case that Carolina does not know the answer is when she has a number 23 with $(2, 7)$ and $(3, 5)$, number 29 with $(2, 9)$ and $(4, 5)$, and number 39 with $(2, 9)$ and $(4, 7)$. Now in the second round, everyone knows the answer, which means the pair is $(2, 9)$, the common pair Ben and Carolina are both unsure.