Northwestern Ontario Annual High School Math Contest

(1) Consider the following sequence:

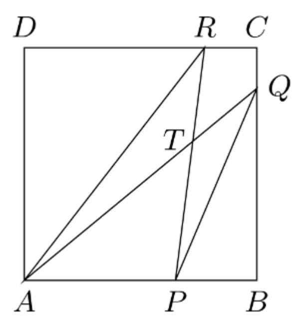
$$a_1 = 86, \qquad a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ even,} \\ a_n + 1 & \text{if } a_n \text{ odd.} \end{cases}$$

Find a_{2021}.

(2) How many positive integers m are such that the sum of all divisors of m (including 1 and m) is equal to $m + 7$?

(3) The class of Alice and Bob has 15 students in total. Desks are arranged in 5 rows of 3 desks each. Students are then randomly assigned to a desk. What is the probability that both Alice and Bob are in the same row?

(4) The square $ABCD$ has side length 30, while $BP = 6$, $CQ = 3$, $CR = 4$. Find the difference between the areas of the triangles $\triangle ART$ and $\triangle PQT$.



(5) What is the 13th largest divisor of 90^{10}?

(6) There are 7 cards with one of the numbers 1, 2, 3, 4, 5, 6, 7 on each. Adam chooses 3 numbers for himself and 3 numbers for Ben. Adam shows his card, and then Ben shows his first card. If the sum of 2 cards is a multiple of 3, Ben wins the game. If the sum is not a multiple of 3, then Adam shows his second card, and so does Ben. Ben wins if the sum of 4 cards is a multiple of 3. Otherwise, they show their third cards. If the sum of 6 cards is a multiple of 3, Ben wins the game. Otherwise, Adam wins. It is known that if Adam chooses the numbers and shows his cards appropriately, he is guaranteed to win, regardless of what Ben does. Describe such a winning strategy.

(7) Find a polynomial with integer coefficients of degree 4 having $\sqrt{2} + \sqrt{3}$ as a root.

(8) List all three digit integers $\underline{abc}$, with $a, c \neq 0$, such that $\underline{abc} - \underline{cba}$ is a multiple of 7.

(9) Solve the equation $2(x+y)^2 - 2(x+y) + 2(x+y)y + y^2 + 1 = 0$.

(10) Among pairs (a, b) of real numbers such that $|a+b| + |a-b| = 1$, what is the largest possible value of $a^2 + b^2 + 2a$?

(11) How many pairs (m, n) of integers are solutions to the equation $m^2 - 4m + n^2 = 9$?

(12) A triangle in the plane has side lengths $5, 12$, and 13. A circle is drawn in such a way that each of the vertices of the triangle lies on the circle. What is the area of the disk bounded by the circle?

(13) Find all solutions θ, with $0 \leq \theta \leq \pi$, to the equation $\sqrt{1 + \sin\theta} - \sqrt{1 - \sin\theta} = 1$.