Northwestern Ontario Annual High School Math Contest

(1) Consider the following sequence:

$$a_1 = 86, \qquad a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ even,} \\ a_n + 1 & \text{if } a_n \text{ odd.} \end{cases}$$

Find a_{2021}.

Solution. We can find that the first 11 terms are 86, 43, 44, 22, 11, 12, 6, 3, 4, 2, 1. Then the next term is $a_{12} = 2$, and then again $a_{13} = 1$, so $a_{2021} = 1$.

(2) How many positive integers m are such that the sum of all divisors of m (including 1 and m) is equal to $m + 7$?

Solution. Write

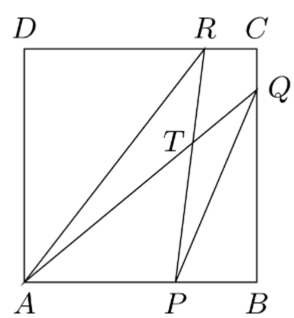
$$m + 7 = m + \text{proper divisors of } m,$$

so we need to write 7 as as sum of different positive integers: there are only two such ways, i.e. $7 = 1 + 6$, which gives no solution since there is no integer m with only divisors $1, 6, m$, and $7 = 1 + 2 + 4$, which gives the only solution $m = 8$.

(3) The class of Alice and Bob has 15 students in total. Desks are arranged in 5 rows of 3 desks each. Students are then randomly assigned to a desk. What is the probability that both Alice and Bob are in the same row?

Solution. Alice can be in any place, while for Bob to be in the same row, he must be assigned to the remaining 2 desks in Alice's row, $2/14 = 1/7$ probability.

(4) The square $ABCD$ has side length 30, while $BP = 6$, $CQ = 3$, $CR = 4$. Find the difference between the areas of the triangles $\triangle ART$ and $\triangle PQT$.



Solution. Note that $AP = 24$, so the area of $\triangle ARP$ is 360, $BQ = 27$, so the area of $\triangle AQP$ is 324. Then, since $\triangle ATP$ is common, the difference between the areas of the triangles $\triangle ART$ and $\triangle PQT$ is just 36.

(5) What is the 13th largest divisor of 90^{10}?

Solution. $90 = 2 \cdot 3^2 \cdot 5$, so $90^{10} = 2^{10} \cdot 3^{20} \cdot 5^{10}$, and the 13 smallest divisors of 90^{10} are $1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18$. Thus the 13th largest is $90^{10}/18 = 2^9 \cdot 3^{18} \cdot 5^{10}$.

(6) There are 7 cards with one of the numbers 1, 2, 3, 4, 5, 6, 7 on each. Adam chooses 3 numbers for himself and 3 numbers for Ben. Adam shows his card, and then Ben shows his first card. If the sum of 2 cards is a multiple of 3, Ben wins the game. If the sum is not a multiple of 3, then Adam shows his second card, and so does Ben. Ben wins if the sum of 4 cards is a multiple of 3. Otherwise, they show their third cards. If the sum of 6 cards is a multiple of 3, Ben wins the game. Otherwise, Adam wins. It is known that if Adam chooses the numbers and shows his cards appropriately, he is guaranteed to win, regardless of what Ben does. Describe such a winning strategy.

Solution. Adam gives Ben 1, 4, 7, while keeping 2, 3, 5 for himself. The key is that, since Ben's cards are all of the form $3a+1$, i.e. multiple of 3, plus 1, Adam can control the sum (modulus 3). Adam first shows the card 3: independently of what Ben shows, the sum will be either 4, 7, or 10, in all cases a multiple of 3, plus 1. So Adam shows the second card, which can be either 2 or 5: for simplicity, let Adam show 2, so the total sum of the 3 cards shown is now a multiple of 3. Regardless of which card Ben shows, the sum of these 4 cards becomes again a multiple of 3, plus 1. Finally, Adam shows 5 (hence the sum of the 5 cards shown until now will be a multiple of 3), and Ben shows his last card. The total sum of these 6 cards will be a multiple of 3, plus 1. Thus Adam wins.

(7) Find a polynomial with integer coefficients of degree 4 having $\sqrt{2}+\sqrt{3}$ as a root.

Solution. Clearly, $x-(\sqrt{2}+\sqrt{3})$ has $\sqrt{2}+\sqrt{3}$ as a root. Multiplying by $(x-(\sqrt{2}-\sqrt{3}))$ gives

$$(x-(\sqrt{2}+\sqrt{3}))(x-(\sqrt{2}-\sqrt{3})) = x^2-2\sqrt{2}x-1,$$

and multiplying again by $x^2+2\sqrt{2}x-1$ gives

$$(x^2-2\sqrt{2}x-1)(x^2+2\sqrt{2}x-1) = x^4-10x^2+1,$$

which is an acceptable solution.

(8) List all three digit integers $\underline{abc}$, with $a, c \neq 0$, such that $\underline{abc}-\underline{cba}$ is a multiple of 7.

Solution. For $\underline{abc}-\underline{cba}$ to be a multiple of 7, we need

$$100(a-c)+c-a = 99(a-c)$$

to be a multiple of 7. Since 99 is not, we need $(a-c)$ to be a multiple of 7. The only possible choices are thus $a=9$, $c=2$, and $a=8$, $c=1$. Thus the answers are numbers of the form $\underline{9b2}$, $\underline{8b1}$, with $b=0,\cdots,9$.

(9) Solve the equation $2(x+y)^2-2(x+y)+2(x+y)y+y^2+1=0$.

Solution. Rewrite as

$$\begin{aligned} &2(x+y)^2-2(x+y)+2(x+y)y+y^2+1 \\ &= \underbrace{(x+y)^2-2(x+y)+1}_{=(x+y-1)^2}+\underbrace{(x+y)^2+2(x+y)y+y^2}_{=(x+2y)^2}=0, \end{aligned}$$

hence we need

$$\begin{cases} x+y &= 1, \\ x+2y &= 0, \end{cases}$$

whose only solutions are then $x=2$, $y=-1$.

(10) Among pairs (a,b) of real numbers such that $|a+b|+|a-b|=1$, what is the largest possible value of a^2+b^2+2a?

Solution. Taking the square both sides gives

$$(|a+b|+|a-b|)^2 = 2(a^2+b^2+|a^2-b^2|) = 1.$$

Here two cases can arise:

- if $a^2 \geq b^2$, then this gives $4a^2 = 1$, and hence $a = \pm 1/2$. Since, for any b, $a^2 + b^2 + 2a$ is larger when taking $a = 1/2$ rather then $a = -1/2$, we take $a = 1/2$. Then, we need to take the largest possible b that satisfies $|1/2 + b| + |1/2 - b| = 1$, which is clearly $b = \pm 1/2$. So, by taking $a = 1/2$, $b = \pm 1/2$, we get
$$a^2 + b^2 + 2a = \frac{3}{2}.$$
- If $a^2 \leq b^2$, then this gives $4b^2 = 1$, and hence $b = \pm 1/2$. In either case, to satisfy $|a + b| + |a - b| = 1$ with the largest possible a, we need to take $a = \pm 1/2$. Since again $a^2 + b^2 + 2a$ is larger when taking $a = 1/2$ rather then $a = -1/2$, we take $a = 1/2$. Thus again by taking $a = 1/2$, $b = \pm 1/2$, we get
$$a^2 + b^2 + 2a = \frac{3}{2}.$$

So the maximum possible value for $a^2 + b^2 + 2a$ is $3/2$.

(11) How many pairs (m, n) of integers are solutions to the equation $m^2 - 4m + n^2 = 9$?
Solution. Adding 4 both sides gives
$$m^2 - 4m + 4 + n^2 = (m - 2)^2 + n^2 = 13.$$
The only way to write 13 as a sum of squares is $13 = 4 + 9 = (\pm 2)^2 + (\pm 3)^2$, so we need either $m - 2 = \pm 2$, $n = \pm 3$ (4 choices), or $m - 2 = \pm 3$, $n = \pm 2$ (again 4 choices), for a total of 8 choices.

(12) A triangle in the plane has side lengths $5, 12$, and 13. A circle is drawn in such a way that each of the vertices of the triangle lies on the circle. What is the area of the disk bounded by the circle?
Solution. Note that $5, 12$, and 13 form a Pythagorean triple, so the triangle has a right angle, and hypotenuse of length 13. The circle across its vertices is thus the circumscribed circle, whose diameter is the hypotenuse, whose length is 13. Such disk clearly has area $\pi \times 13^2/4$.

(13) Find all solutions θ, with $0 \leq \theta \leq \pi$, to the equation $\sqrt{1 + \sin\theta} - \sqrt{1 - \sin\theta} = 1$.
Solution. Taking the squares both sides gives
$$2 - 2\sqrt{1 + \sin\theta}\sqrt{1 - \sin\theta} = 2(1 - \sqrt{1 - \sin^2\theta}) = 2(1 - |\cos\theta|) = 1,$$
hence we need $\cos\theta = \pm 1/2$, i.e. $\theta = \pi/3, 2\pi/3$.